

On integrable directed polymer models on the square lattice

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In a recent work Povolotsky²⁴ provided a three-parameter family of stochastic particle systems with zero-range interactions in one dimension which are integrable by coordinate Bethe ansatz. Using these results we obtain the corresponding condition for integrability of a class of directed polymer models with random weights on the square lattice. Analyzing the solutions we find, besides known cases, a new two-parameter family of integrable DP model, which we call *the Inverse-Beta polymer*, and provide its Bethe ansatz solution.

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I. INTRODUCTION AND MAIN RESULTS

A. overview

There is considerable recent interest in exact solutions for models in the universality class of the 1D stochastic growth Kardar-Parisi-Zhang equation (KPZ)¹. Models in the KPZ class share the same large time statistics, also found to be related to the universal statistics of large random matrices². Methods developed in the context of quantum integrability are exploited and broadly extended to solve a variety of 1D stochastic models. The Bethe ansatz solution of the attractive delta Bose gas (the Lieb-Liniger model^{3,4}) was combined with the replica method⁵, to obtain exact solutions for the KPZ equation directly in the continuum and at arbitrary time, for the main classes of initial conditions (droplet, flat, stationary, half-space)^{6–15}. The Cole-Hopf mapping $h \sim \ln Z$ is used, where h is the height of the KPZ interface and Z the partition sum of a directed polymer in a random potential (DP). Hence in the continuum, studying KPZ growth is equivalent to studying the DP model, an equilibrium statistical mechanics problem with quenched disorder. The time in KPZ growth becomes the length of the polymer t . The replica Bethe ansatz (RBA) method then allows to calculate the integer moments $\overline{Z^n}$ and, from them, to retrieve the probability distribution function (PDF) of Z . Since the last step is non-rigorous because of the fast growth of these moments, the mathematical community has concentrated on the exact solution of discrete models, which in favorable cases, do not suffer from the moment growth problem. Discrete models, such as the PNG growth model^{16–18}, the TASEP and ASEP particle transport model^{19,20} and discrete DP models^{17,21,22} played a pioneering role in unveiling the universal statistics of the KPZ class at large time (the Airy processes). Recently they have been considerably generalized, unveiling a very rich underlying "stochastic integrability" structure^{23–33}. Since in suitable limits (e.g. ASEP with weak asymmetry, q -TASEP with $q \rightarrow 1$, semi-discrete DP) they converge to the continuum KPZ equation, they also led to some recent rigorous results for KPZ at arbitrary time^{35–39}.

Besides their interest in relation to KPZ growth, directed polymers are also important in a variety of fields. This includes optimization and glasses^{40,41}, vortex lines in superconductors⁴², domain walls in magnets⁴³, disordered conductors⁴⁴, Burgers equation in fluid mechanics⁴⁵, exploration-exploitation tradeoff in population dynamics and economics⁴⁶ and in biophysics^{47,48}. In some situations (heavy tailed disorder) they exhibit anomalous (non-standard KPZ) scaling^{49–51}. Apart from models on trees, exactly solvable models of DP (e.g. on regular lattices) remain, however, exceedingly rare. We will present in this paper a new solvable DP model.

On the square lattice a few remarkable solvable DP models have been found. The first that was discovered is at zero temperature $T = 0$ (i.e. it amounts to find the minimal energy path, energies being additive along a path), with a geometric distribution (of parameter $q < 1$) of on-site random potentials²¹. The second that was discovered, called the log-Gamma polymer⁵², is a finite temperature model (as it focuses on Boltzmann weights, which are multiplicative along a path), with a so-called inverse gamma distribution for the on-site random weights, with parameter γ . This weight distribution has the peculiarity of exhibiting a fat tail $P(w) \sim w^{-1+\gamma}$. These models are not unrelated: in the limit $\gamma \rightarrow 0$ (so-called zero temperature) the log-Gamma converges to the $q \rightarrow 1$ limit of the Johansson model (i.e. with exponentially distributed on-site weights)⁵³. They were both proved to belong to the KPZ class, with convergence of the free energy PDF to the GUE Tracy Widom distribution. The Johansson model was solved as a determinantal process²¹. The log-Gamma model was solved using the gRSK correspondence (a geometric lifting of the Robinson-Schensted-Knuth (RSK) correspondence) leading finally to an expression for the Laplace transform of $P(Z)$ as a Fredholm determinant^{53,54}.

Recently, we provided a solution of the log-Gamma polymer using replicas and the coordinate Bethe ansatz, closer in spirit to the integrability methods used to solve KPZ⁵⁵. As in the continuum, this replica Bethe ansatz approach

consists in computing the moments of the partition sum of the DP using a transfer matrix (i.e. recursive) formulation of the problem. This formulation can formally be interpreted as a discrete-time quantum mechanical model of interacting Bosons. Such a connection between discrete-time particle models and lattice DP was also noted, and exploited in⁵⁶ to unveil and study a new integrable DP model with Gamma distributed Boltzmann weights, called the “Strict-Weak” DP model, as the $q \rightarrow 1$ limit of the discrete time q -TASEP model⁵⁸. In parallel, this DP model was also solved using the gRSK correspondence in⁵⁷.

In a recent seminal work, Povolotsky²⁴ provided a three parameter family of discrete-time stochastic interacting particles systems with zero range interactions (“zero-range processes” (ZRP)) called the (q, μ, ν) -Boson process and integrable by coordinate Bethe ansatz. This led to further rigorous work on this class of particle model and on a dual model, termed the q -Hahn TASEP, which eventually allowed to unify integrability properties of ASEP and q -TASEP, a long-standing goal^{25,30,31}. On the directed polymer side, this work also led to the discovery of a new integrable model, called the “Beta” polymer, introduced and studied in⁵⁹. There the model was solved as a $q \rightarrow 1$ limit of the (q, μ, ν) -Boson (in analogy with the Strict-Weak case), but the authors also already provided a direct replicas Bethe ansatz solution of the model.

The aim of the present paper is to explore more systematically the consequences of Povolotsky’s work to directly search for, and attempt to classify, the corresponding family of integrable DP models. Integrability then leads to a constraint on the integer moments of the Boltzmann weights distributions, and we search for solutions in terms of PDF of bond and site disorder.

We find that there are two main solutions, the first one corresponding to the Beta polymer⁵⁹. The second however is new and corresponds to weights v on horizontal bonds, and u on vertical bonds of the square lattice, with the following PDF: u , is distributed according to:

$$\tilde{p}_{\gamma,\beta}(u) = \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1}, \quad u \in [1, +\infty[\quad , \gamma, \beta > 0 \quad (1)$$

The weights are correlated on bonds which share a top/right site (see Fig. 1), with $v = u - 1 \in [0, +\infty[$ but otherwise uncorrelated. Given the form of (1) we call our new model the Inverse-Beta polymer⁶³.

We will provide in this paper the coordinate Bethe ansatz solution to this model, as well as some explicit integral representation and Fredholm determinant formulas for its Laplace transform. It is interesting to note that for $\beta \rightarrow +\infty$ this model, under suitable rescaling, converges to the log-Gamma polymer (see below). Hence it can be considered as a generalization of the log-Gamma polymer.

B. Main results and outline of the paper

The first result of this paper, obtained in Section II, are some general conditions for a finite temperature model of directed polymer on the square lattice to be integrable using the coordinate Bethe ansatz. The only hypothesis are that Boltzmann weights on horizontal edges and vertical edges can be correlated only if they share the same top or right site (an example of short-range correlations), and that they are homogeneously distributed. Within this framework, in Section III, we attempt a classification of integrable DP models, retrieve the known integrable models and introduce a new one, the Inverse-Beta polymer, whose Boltzmann weights are distributed as (1). This model has two parameters $\gamma, \beta > 0$ and contains the log-Gamma and Strict-Weak polymers as scaling limits. More precisely, we show that the partition sum $Z_t(x)$ of the Inverse-Beta model (see Section II A for the definition) converges in law to the partition sum of the log-Gamma (resp. Strict-Weak) polymer $Z_t^{LG}(x)$ (resp. $Z_t^{SW}(x)$) as

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta^t} Z_t(x) \sim Z_t^{LG}(x) \quad , \quad \lim_{\gamma \rightarrow \infty} \gamma^x Z_t(x) \sim Z_t^{SW}(x) . \quad (2)$$

In Section IV we use the coordinate Bethe ansatz to study the Inverse-Beta polymer with point-to-point boundary conditions. We obtain an exact result for the integer moments of the partition sum $Z_t(x)^n$ (60), defined for $n < \gamma$. Using this result, we conjecture the formula (69) that expresses the Laplace transform of $Z_t(x)$ as a Fredholm

⁶³Note that a nomenclature based on the names of the weight distributions, the log-Gamma polymer could be called the Inverse-Gamma polymer, and the Strict-Weak the Gamma polymer. Alternatively our model could be called the log-Beta polymer.

determinant $\overline{e^{-uZ_t(x)}} = \text{Det}(I + K_{tx})$ with

$$K_{t,x}(v_1, v_2) = \int_{-\infty}^{+\infty} \frac{dk}{\pi} \frac{-1}{2i} \int_C \frac{ds}{\sin(\pi s)} u^s e^{-2ik(v_1-v_2)-s(v_1+v_2)} \quad (3)$$

$$\left(\frac{\Gamma(-\frac{s}{2} + \frac{\gamma}{2} - ik)}{\Gamma(\frac{s}{2} + \frac{\gamma}{2} - ik)} \right)^{1+x} \left(\frac{\Gamma(-\frac{s}{2} + \frac{\gamma}{2} + ik)}{\Gamma(\frac{s}{2} + \frac{\gamma}{2} + ik)} \right)^{1-x+t} \left(\frac{\Gamma(\beta + ik + \frac{\gamma}{2} + \frac{s}{2})}{\Gamma(\beta + ik + \frac{\gamma}{2} - \frac{s}{2})} \right)^t$$

where $C = a + i\mathbb{R}$ with $0 < a < \min(1, \gamma)$ and $K_{t,x} : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$. Alternatively, we obtain an equivalent Fredholm determinant for the same quantity with a different kernel (which contains notably one less integral) in (70). By analogy with a known formula for the log-Gamma polymer, we also conjecture a n-fold integral formula (71) for the Laplace transform

$$\overline{e^{-uZ_t(x)}} = \frac{1}{J!} \int_{(iR)^J} \prod_{j=1}^J \frac{dw_j}{2i\pi} \prod_{j \neq k=1}^J \frac{1}{\Gamma(w_j - w_k)} \quad (4)$$

$$\left(\prod_{j=1}^J u^{w_j - a} \Gamma[a - w_j]^J \left(\frac{\Gamma(\gamma + a - w_j)}{\Gamma(\gamma)} \right)^I \left(\frac{\Gamma(w_j - a + \beta)}{\Gamma(\beta)} \right)^{I+J-2} \right),$$

with $0 < a < \min(1, \gamma)$, valid for $\text{Re}(u) > 0$, $1 \leq J \leq I$ and where $x = I - 1$ and $t = I + J - 2$. Using an asymptotic analysis of our Fredholm determinant formulas, we show in Section IV C the KPZ universality of the model for polymers of large length $t \rightarrow \infty$ with an arbitrary angle $\varphi \in]-1/2, 1/2[$ with respect to the diagonal. More precisely, we show

$$\lim_{t \rightarrow \infty} \text{Prob} \left(\frac{\log Z_t((1/2 + \varphi)t) + tc_\varphi}{\lambda_\varphi} < 2^{\frac{2}{3}} z \right) = F_2(z) \quad (5)$$

where $F_2(z)$ is the standard GUE Tracy-Widom cumulative distribution function, $\lambda_\varphi \sim t^{1/3}$ and the (φ -dependent) constants are determined by a system of equations (80) that involves the digamma function ψ . As a particular case we study these characteristic constants for long polymers with the ‘optimal angle’ $\varphi = \varphi^*$ (in the sense that the mean free energy c_φ is minimal for this angle) and find explicit expressions as

$$\varphi^* = -\frac{1}{2} \frac{\psi'(\beta + \gamma/2)}{\psi'(\gamma/2)} < 0$$

$$c_{\varphi^*} = \psi(\gamma/2) - \psi(\beta + \gamma/2)$$

$$\lambda_{\varphi^*} = \left(\frac{t}{8} (\psi''(\beta + \gamma/2) - \psi''(\gamma/2)) \right)^{1/3}. \quad (6)$$

Finally, in Section (IV D) we study a two parameters zero temperature DP model that we obtain as the limit $\gamma = \epsilon\gamma'$ and $\beta = \epsilon\beta'$ with $\epsilon \rightarrow 0$ of the Inverse-Beta polymer. This study is close in spirit to the one made in⁵⁹ where the zero temperature limit of the Beta polymer is studied, but the models are qualitatively very different. The energy of this model are distributed as $(\mathcal{E}'_u, \mathcal{E}'_v) = (-\zeta E_{\gamma'}, (1 - \zeta)E_{\beta'} - \zeta E_{\gamma'})$ where $(\mathcal{E}'_u, \mathcal{E}'_v)$ are the energies on vertical and horizontal edges, ζ is a Bernoulli random variable of parameter $p = \beta' / (\gamma' + \beta')$ and $E_{\gamma'}$ and $E_{\beta'}$ are exponential random variables of parameter $\gamma' > 0$ and $\beta' > 0$, independent of ζ . This model generalizes the known zero temperature limit of the log-Gamma directed polymer and we obtain exact results for the cumulative distribution of the optimal energy, noted $\mathfrak{E}_{(t,x)}$, of this zero temperature model $\text{Prob}(\mathfrak{E}_{(t,x)} > r)$. In particular we obtain a Fredholm determinant formula (99) $\text{Prob}(\mathfrak{E}_{(t,x)} > r) = \text{Det}(I + K_{tx}^{T=0})$ with

$$K_{t,x}^{T=0}(v_1, v_2) = - \int_{-\infty}^{+\infty} \frac{dk}{\pi} \int_C \frac{ds}{2i\pi s} e^{sr - 2ik(v_1-v_2) - s(v_1+v_2)} \quad (7)$$

$$\left(\frac{\frac{s}{2} + \frac{\gamma'}{2} - ik}{-\frac{s}{2} + \frac{\gamma'}{2} - ik} \right)^{1+x} \left(\frac{\frac{s}{2} + \frac{\gamma'}{2} + ik}{-\frac{s}{2} + \frac{\gamma'}{2} + ik} \right)^{1-x+t} \left(\frac{\beta' + ik + \frac{\gamma'}{2} - \frac{s}{2}}{\beta' + ik + \frac{\gamma'}{2} + \frac{s}{2}} \right)^t.$$

where $\tilde{C} = a + i\mathbb{R}$ with $0 < a < \gamma'$ and $K_{t,x}^{T=0} : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$. We also conjecture an equivalent n-fold integral formula (101)

$$\text{Prob}(\mathfrak{E}_{(t,x)} > r) = \frac{1}{J!} \int_{(iR)^J} \prod_{j=1}^J \frac{dw_j}{2i\pi} \prod_{j \neq k=1}^J (w_j - w_k) \prod_{j=1}^J \frac{e^{r(w_j - a)}}{(a - w_j)^J} \left(\frac{\gamma'}{\gamma' + a - w_j} \right)^I \left(\frac{\beta'}{w_j - a + \beta'} \right)^{I+J-2}. \quad (8)$$

with $0 < a < \gamma'$. Using our exact results, we conclude this section by showing the KPZ universality of the zero temperature model in (106). In the case $\beta'/\gamma' \rightarrow \infty$ the model maps onto the Johansson DP model with an exponential distribution and we show that our solution reproduces all the (non trivial) angle dependent normalizing constants in the statement of convergence to the GUE Tracy Widom distribution.

A series of appendices also contains additional discussions and some technical details separated from the main text for clarity.

II. DIRECTED POLYMERS ON THE SQUARE LATTICE: REPLICA METHOD AND INTEGRABILITY

A. Definition of the model

We consider the square lattice \mathbb{Z}^2 with coordinates (t, x) with x the usual horizontal coordinate and t a coordinate⁶⁴ running through the diagonal of \mathbb{Z}^2 as depicted in Fig. 1. We will also sometime use the usual euclidean coordinates (I, J) on \mathbb{Z}^2 with $x = I - 1$ and $t = I + J - 2$. The first quadrant is thus $I, J \geq 1$ and $x, t \geq 0$. A directed polymer model on \mathbb{Z}^2 is defined by the partition sum

$$Z_t(x) = \sum_{\pi: (0,0) \rightarrow (t,x)} \prod_{e \in \pi} w_e \quad (9)$$

where the sum is on all directed (i.e. up/right) paths with fixed starting point $(0, 0)$ (corresponding to $(I, J) = (1, 1)$) and endpoint (t, x) , and the product is on all edges $e = (t', x') \rightarrow (t' + 1, x')$ or $e = (t', x') \rightarrow (t' + 1, x' + 1)$ visited by π . Here for definiteness we consider a directed polymer model with fixed endpoints, but the model can be generalized to other boundary conditions. We also restrict ourselves to models with on-links Boltzmann weights w_e . Obviously, by redefining the weights w_e , one can also include on-sites Boltzmann weights so that this hypothesis is non restrictive. The Boltzmann weights are positive random variables $w_e \in \mathbb{R}_+$. We will generally note u (resp. v) the Boltzmann weights on vertical (resp. horizontal) edges:

$$\begin{aligned} w_e &= u_{t,x} \text{ if } e = (t-1, x) \rightarrow (t, x) , \\ w_e &= v_{t,x} \text{ if } e = (t-1, x-1) \rightarrow (t, x) . \end{aligned} \quad (10)$$

We will consider the class of models with the following structure of local correlations, which naturally emerges in the integrable family we are studying: the weights on edges arriving at different sites are statistically independent, but the weights of two edges arriving at the same site are correlated. Thus the model is defined by a (common) joint PDF for the weights of the type $(u_{t,x}, v_{t,x})$, denoted $p(u, v)$ (see Fig. 1). The pairs $(u_{t,x}, v_{t,x})$ are chosen independently from site to site. In the following, the overline $\overline{(\cdot)}$ denotes the average of a quantity over all realizations for the $(u_{t,x}, v_{t,x})$.

B. The replica method and the coordinate Bethe Ansatz.

In general, one is interested in computing the PDF of $Z_t(x)$ or of its logarithm $\log Z_t(x)$. The replica method consists in first studying the equal-time moments of $Z_t(x)$: for $n \in \mathbb{N}$ and whenever they exist, one defines

$$\psi_t(x_1 \dots, x_n) = \overline{Z_t(x_1) \dots Z_t(x_n)} . \quad (11)$$

In the general case these moments are only defined for $n \leq n_{\max}$, because of possible fat-tail in the distribution of Boltzmann weights, such that $\overline{u^{n_1} v^{n_2}}$ are finite for $n_1 + n_2 \leq n_{\max}$ but are infinite for some (n_1, n_2) with $n_1 + n_2 > n_{\max}$. In the log-Gamma case one has $n_{\max} = \lfloor \gamma^- \rfloor$ whereas in the Beta polymer case $n_{\max} = \infty$. Obtaining the PDF of $Z_t(x)$ from the knowledge of the moments is usually non-trivial, especially when $n_{\max} < \infty$. In this case the procedure is non-rigorous and one has to perform some analytical continuation as in the log-Gamma case. In most of this section

⁶⁴Note that the (t, x) coordinates of the present paper do not coincide with the ones of⁵⁵ that we denote (T, X) . To compare formulas, one can use $t = T$ and $x = t/2 + X$.

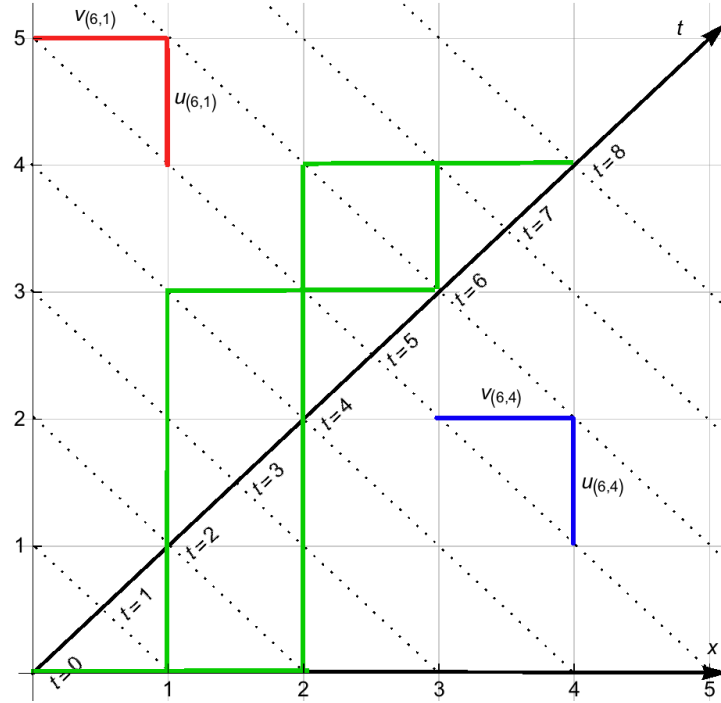


FIG. 1: General scheme for the models of directed polymer studied here. Blue (resp. Red) : couple of correlated Boltzmann weight on edges arriving at $(t = 6, x = 1)$ (resp. $(t = 6, x = 4)$). Green: two admissible (i.e. up/right) paths for polymers with starting point $(0, 0)$ and endpoint $(8, 4)$.

we will not discuss this issue and only focus on computing the moments for $n \leq n_{\max}$, which is a well-defined problem. The problem of computing ψ_t is manageable thanks to a the recursive formulation of (9):

$$\begin{aligned} Z_{t=0}(x) &= \delta_{x,0} \\ Z_{t+1}(x) &= u_{t+1,x} Z_t(x) + v_{t+1,x} Z_t(x-1) . \end{aligned} \quad (12)$$

This can be translated to a recursive (i.e. transfer matrix) equation for ψ_t :

$$\begin{aligned} \psi_{t=0}(x_1 \dots, x_n) &= \delta_{x_1,0} \dots \delta_{x_n,0} \\ \psi_{t+1}(x_1 \dots, x_n) &= \sum_{\{\delta_1, \dots, \delta_n\} \in \{0,1\}^n} a_{x_1, \dots, x_n}^{\delta_1, \dots, \delta_n} \psi_t(x_1 - \delta_1, \dots, x_n - \delta_n) = (T_n \psi_t)(x_1 \dots, x_n) \\ a_{x_1, \dots, x_n}^{\delta_1, \dots, \delta_n} &= \prod_{y \in \mathbb{Z}} (u)^{\sum_{i=1}^n \delta_{x_i, y} \delta_{\delta_i, 0}} (v)^{\sum_{i=1}^n \delta_{x_i, y} \delta_{\delta_i, 1}} . \end{aligned} \quad (13)$$

Where we used the statistical independence of the Boltzmann weights ending at different sites and the definition of (u, v) . Note that the evolution equation (13) is symmetric by exchange $x_i \leftrightarrow x_j$. Therefore, since the initial condition is also fully symmetric, if one is able to find all the symmetric eigenfunctions ψ_μ of T_n , i.e. a complete basis of symmetric functions such that $T_n \psi_\mu = \Lambda_\mu \psi_\mu$, the problem is essentially solved. In the already known models, it was possible to find the eigenfunctions of T_n in the form of the coordinate Bethe Ansatz. More precisely, in the sector $W_n = \{x_1 \leq \dots \leq x_n\}$ (this defines the Weyl chamber), one looks for eigenfunctions of the form

$$\begin{aligned} \psi_\mu(x_1, \dots, x_n) &= \tilde{\psi}_\mu(x_1, \dots, x_n) \text{ if } x_1 \leq \dots \leq x_n \\ \tilde{\psi}_\mu(x_1, \dots, x_n) &= \sum_{\sigma \in S_n} A_\sigma \prod_{i=1}^n z_{\sigma(i)}^{x_i} \end{aligned} \quad (14)$$

where the sum runs over all permutations of $\{1, \dots, n\}$, and the variables A_σ and z_i are complex numbers. The wave function $\psi_\mu(x_1, \dots, x_n)$ is deduced in the other sectors by using that it is fully symmetric function of its arguments. As one can guess, this form of eigenfunction is restrictive and it can only works if the variables $a_{x_1, \dots, x_n}^{\delta_1, \dots, \delta_n}$, or equivalently

the integer moments $\overline{u^{n_1} v^{n_2}}$ for $(n_1, n_2) \in \mathbb{N}^2$ obey a particular structure. Thus one can hope to classify the models that are solvable by the coordinate Bethe Ansatz. In fact, (13) is reminiscent of the equations usually considered in the study of zero-range stochastic particle systems, for which a classification was proposed in²⁴ and latter extended in³³. In the next section we follow the route of²⁴ and adapt it to our setting to deduce a classification of integrable directed polymers models⁶⁵.

C. The constraint of integrability on integer moments.

If one can diagonalize the evolution equation (13) using the Bethe Ansatz (14), then in the sector $\overset{\circ}{W}_n = \{x_1 < \dots < x_n\}$ (i.e. the interior of the Weyl chamber where all particles sit on distinct sites and do not interact), one must have

$$\begin{aligned} \Lambda_\mu \psi_\mu(x_1, \dots, x_n) &= (T_n \psi_\mu)(x_1, \dots, x_n) \\ \Lambda_\mu \tilde{\psi}_\mu(x_1, \dots, x_n) &= \sum_{\{\delta_1, \dots, \delta_n\} \in \{0,1\}^n} (\bar{u})^{n - \sum_{i=1}^n \delta_i} (\bar{v})^{\sum_{i=1}^n \delta_i} \tilde{\psi}_\mu(x_1 - \delta_1, \dots, x_n - \delta_n) \\ &= \prod_{i=1}^n (\bar{u} + \bar{v} z_i^{-1}) \tilde{\psi}_\mu(x_1, \dots, x_n), \end{aligned} \quad (15)$$

and this already imposes the eigenvalue to be $\Lambda_\mu = \prod_{i=1}^n (\bar{u} + \bar{v} z_i^{-1})$. Note that this is a direct consequence of the weights having zero-range interaction: in the $\overset{\circ}{W}_n$ sector, the operator T_n just acts as a (biased) diffusion operator on the one-dimensional line. Let us now look at what happens when exactly two particles are at the same position: $x_1 < \dots < x_i = x_{i+1} < \dots < x_n$. In this case, the evolution equation reads

$$\begin{aligned} \Lambda_\mu \psi_\mu(x_1, \dots, x_n) &= \sum_{\{\delta_1, \dots, \delta_n\} - \{\delta_i, \delta_{i+1}\} \in \{0,1\}^{n-2}} (\bar{u})^{n-2 - \sum_{j=1, j \neq i, i+1}^n \delta_j} (\bar{v})^{\sum_{j=1, j \neq i, i+1}^n \delta_j} \left(\bar{u}^2 \psi_\mu(x_1 - \delta_1, \dots, x_i, x_i, \dots, x_n - \delta_n) \right) \\ &\quad + 2\bar{u}\bar{v} \psi_\mu(x_1 - \delta_1, \dots, x_i - 1, x_i, \dots, x_n - \delta_n) + \bar{v}^2 \psi_\mu(x_1 - \delta_1, \dots, x_i - 1, x_i - 1, \dots, x_n - \delta_n) \end{aligned} \quad (16)$$

Where we used the symmetry of ψ_μ to express each terms with coordinates in the Weyl chamber W_n . However, the left-hand side of (16) is already constrained to be equal to the right-hand side (last line) of (15) even for $x_i = x_{i+1}$ because the eigenvalue Λ_μ is entirely determined by (15). For this equality to hold $\forall x_1 < \dots < x_i = x_{i+1} < \dots < x_n$ for an eigenfunction of the form (14) one must have, rewriting (16) in terms of $\tilde{\psi}_\mu$,

$$\begin{aligned} &\bar{u}^2 \tilde{\psi}_\mu(x_1 - \delta_1, \dots, x_i, x_i, \dots, x_n - \delta_n) + 2\bar{u}\bar{v} \tilde{\psi}_\mu(x_1 - \delta_1, \dots, x_i - 1, x_i, \dots, x_n - \delta_n) \\ &\quad + \bar{v}^2 \tilde{\psi}_\mu(x_1 - \delta_1, \dots, x_i - 1, x_i - 1, \dots, x_n - \delta_n) \\ &= (\bar{v})^2 \tilde{\psi}_\mu(x_1 - \delta_1, \dots, x_i - 1, x_i - 1, \dots, x_n - \delta_n) + (\bar{v})(\bar{u}) \tilde{\psi}_\mu(x_1 - \delta_1, \dots, x_i - 1, x_i, \dots, x_n - \delta_n) \\ &\quad + (\bar{u})(\bar{v}) \tilde{\psi}_\mu(x_1 - \delta_1, \dots, x_i, x_i - 1, \dots, x_n - \delta_n) + (\bar{u})^2 \tilde{\psi}_\mu(x_1 - \delta_1, \dots, x_i, x_i, \dots, x_n - \delta_n). \end{aligned} \quad (17)$$

Notice that the third term in the right-hand side in (17) involves coordinates outside of the Weyl chamber and is thus not a physical term. For the two particles problem to be solved, it must have the value

$$\begin{aligned} \tilde{\psi}_\mu(x_i, x_i - 1) &= a \tilde{\psi}_\mu(x_i, x_i) + b \tilde{\psi}_\mu(x_i - 1, x_i) + c \tilde{\psi}_\mu(x_i - 1, x_i - 1) \\ a &= \frac{\bar{u}^2 - (\bar{u})^2}{(\bar{u})(\bar{v})} \quad b = \frac{2\bar{u}\bar{v} - (\bar{u})(\bar{v})}{(\bar{u})(\bar{v})} \quad c = \frac{\bar{v}^2 - (\bar{v})^2}{(\bar{u})(\bar{v})}, \end{aligned} \quad (18)$$

where here, for clarity, we did not write the other particles positions. For obvious reasons, (18) is called the two-particles boundary condition. It is also a consequence of the short-ranged nature of the correlations between Boltzmann weights that the two-particles evolution equation can simply be interpreted as a two particles boundary conditions. In

⁶⁵Note that the extension of³³ of the classification of²⁴ corresponds to stochastic particles systems with non-simultaneous updates. Hence the classification of²⁴ is sufficient in our context.

terms of the Bethe Ansatz (14), it imposes the quotient of two amplitudes A_σ related to each other by a permutation to be given by

$$S(z_i, z_j) := \frac{A_{\dots j i \dots}}{A_{\dots i j \dots}} = -\frac{c + b z_j + a z_i z_j - z_i}{c + b z_i + a z_i z_j - z_j}. \quad (19)$$

Where this defines the S matrix. This can be solved as

$$A_\sigma = \epsilon(\sigma) \prod_{1 \leq i < j \leq n} \frac{c + b z_{\sigma(i)} + a z_{\sigma(i)} z_{\sigma(j)} - z_{\sigma(j)}}{c + b z_i + a z_i z_j - z_j}. \quad (20)$$

As a consequence, up to a multiplicative factor, the form of the Bethe Ansatz is now entirely specified and something special has to happen if it also solves the m particles problem (case where m particles are at the same position) for arbitrary $2 \leq m \leq n$. Indeed, for arbitrary m , one can repeat the same analysis and check that the ansatz (14) works, i.e. that the evolution equation with m particles at the same position (generalization of (16)) can be transformed into the free evolution equation (generalization of (17)) by only applying (18) recursively. Schematically, this is conveniently encoded in a non-commutative algebras with two generators (A, B) such that

$$BA = aA^2 + bAB + cB^2, \quad (21)$$

which encodes what happen when one transforms a forbidden term of the form $\tilde{\psi}_\mu(\dots, x_i, x_i - 1 \dots)$ into a sum of terms with coordinates in the Weyl chamber. In this language, the model is indeed integrable with the coordinate Bethe Ansatz (14) if and only if

$$(\bar{u}A + \bar{v}B)^m = \sum_{n_1=0}^m \overline{u^{n_1} v^{m-n_1}} C_m^{n_1} A^{n_1} B^{m-n_1}. \quad (22)$$

Where the right-hand side represents the true evolution equation that only contains terms in the Weyl chamber, i.e. in this language, that only contains ordered words of the form $A^{n_1} B^{n_2}$, and the left hand side is the formal free evolution equation, which contains various terms outside the Weyl chamber, i.e. wrongly ordered words. The right-hand side of (22) can be computed using the formula appearing in²⁴. In the context of this paper, only models satisfying the “stochasticity hypothesis”, $a + b + c = 1$ (corresponding to a conservation of probability) were considered. Here in general, this hypothesis has to be relaxed and $a + b + c \neq 1$. This is easily done by a scale transformation⁶⁶, i.e. we introduce new parameters (ρ, a', b', c') and generators (A', B') , such that

$$a = a' \rho, \quad b = b', \quad c = \frac{c'}{\rho} \quad (23)$$

$$A = \frac{A'}{\rho}, \quad B = B', \quad B'A' = a'A'^2 + b'A'B' + c'B'^2 \quad (24)$$

where ρ is chosen such that $a' + b' + c' = 1$, in which case we can use the results of Ref.²⁴ in terms of the new generators (A', B') and parameters (a', b', c') (called there (A, B) and α, β, γ). We obtain

$$\begin{aligned} (\bar{u}A + \bar{v}B)^m &= \left(\frac{\bar{u}}{\rho}A' + \bar{v}B'\right)^m \\ &= \left(\frac{\bar{u}}{\rho} + \bar{v}\right)^m (pA' + (1-p)B')^m \\ &= \left(\frac{\bar{u}}{\rho} + \bar{v}\right)^m \sum_{n_1=0}^m \phi_{q,\mu,\nu}(n_1|m) (A')^{n_1} (B')^{m-n_1} \\ &= \left(\frac{\bar{u}}{\rho} + \bar{v}\right)^m \sum_{n_1=0}^m \rho^{n_1} \phi_{q,\mu,\nu}(n_1|m) A^{n_1} B^{m-n_1} \end{aligned} \quad (25)$$

⁶⁶We thank A.M. Povolotsky for this remark.

where we used the same notations introduced in Ref.²⁴ for the three parameters of the model q, μ, ν and the auxiliary parameter p , so that

$$p = \frac{\bar{u}}{\bar{u} + \rho\bar{v}} \quad , \quad a' = \frac{\nu(1-q)}{1-q\nu} \quad , \quad b' = \frac{q-\nu}{1-q\nu} \quad , \quad c' = \frac{1-q}{1-q\nu} \quad , \quad \mu = p + \nu(1-p)$$

$$\phi_{q,\mu,\nu}(n_1|m) = \mu^{n_1} \frac{(\frac{\nu}{\mu}; q)_{n_1} (\mu; q)_{m-n_1}}{(\nu; q)_m} \frac{(q; q)_m}{(q; q)_{n_1} (q; q)_{m-n_1}} \quad (26)$$

from equations (8) and (26-27) in Ref.²⁴. The q -Pochhammer symbol (used extensively below) is defined as, for $n > 0$:

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \quad , \quad (a; q)_{-n} = \prod_{k=1}^n (1 - aq^{-k})^{-1} \quad (27)$$

and $(a; q)_0 = 1$. For given parameters q, μ, ν the first equation in (26) fixes the scale factor ρ as a function of \bar{u}/\bar{v} .

Comparing this equation with (22) one sees that one must have, $\forall (n_1, n_2)$ such that $n_1 + n_2 \leq n_{\max}$,

$$\overline{u^{n_1} v^{n_2}} = \left(\frac{\bar{u}}{p}\right)^{n_1} \left(\frac{\bar{v}}{1-p}\right)^{n_2} (\mu)^{n_1} \frac{(\frac{\nu}{\mu}; q)_{n_1} (\mu; q)_{n_2}}{(\nu; q)_{n_1+n_2}} \frac{(q; q)_{n_1+n_2}}{(q; q)_{n_1} (q; q)_{n_2}} \frac{1}{C_{n_1+n_2}^{n_1}} \quad (28)$$

We can now explicitly check that for $(n_1, n_2) = (1, 0), (0, 1)$ the r.h.s gives \bar{u} and \bar{v} and that for $(n_1, n_2) = (2, 0), (1, 1), (0, 2)$ it yields second moments compatible with all relations (18) and (23), (26). Thanks to the above construction based on Ref.²⁴ we know that it solves the integrability constraint for all higher positive integer moments with $n_1 + n_2 \leq n_{\max}$.

In this expression, the power-law parts are unimportant (they can be absorbed into a rescaling of the Boltzmann weights which cannot break the integrability of the model). We can now reverse the construction and study a polymer model defined with weights with moments given by

$$\overline{u^{n_1} v^{n_2}} = \frac{(\frac{\nu}{\mu}; q)_{n_1} (\mu; q)_{n_2}}{(\nu; q)_{n_1+n_2}} \frac{(q; q)_{n_1+n_2}}{(q; q)_{n_1} (q; q)_{n_2}} \frac{1}{C_{n_1+n_2}^{n_1}} := \psi_{q,\mu,\nu}(n_1, n_2) \quad , \quad (29)$$

where $(q, \mu, \nu) \in \mathbb{R}^3$ to obtain real Boltzmann weights. This model is automatically integrable, and with the hypothesis that we made, it is the only form for the moments that leads to integrability. However, we now need to check if this DP model really exists, namely that (29) corresponds to the moments of a PDF $p(u, v)$.

Let us now define the moment problem that we must now solve. We are interested in finding a joint PDF $p(u, v)$ with positive integer moments given by (29) and random variables (u, v) living in one of the four quadrants $(\mathbb{R}^\pm, \mathbb{R}^\pm)$. Indeed, if that is the case we automatically find, using a change of the type $(u, v) \rightarrow (\pm u, \pm v)$, positive random variables with moments given by (29) (eventually multiplied by additional power laws $(\pm 1)^{n_1} (\pm 1)^{n_2}$ which do not spoil integrability). Since we extend our search to also include polymer models with $n_{\max} < \infty$, we will generally look for PDF with moments given by (29) for $n_1 + n_2 \leq n_{\max}$ for some n_{\max} . Note that if $n_{\max} < \infty$, this replica Bethe ansatz method allows us to compute a-priori only a few integer moments of $Z_t(x)$. The ultimate goal of computing the PDF from this knowledge is not a mathematically well-posed problem. Fortunately, as e.g. in the case of the log-Gamma polymer (see⁵⁵ for more details on this issue), and in the case of the Inverse-Beta polymer studied below in this paper, the situation turns out to be more favorable. Indeed in these cases, though $n_{\max} < \infty$, the complex moments $\overline{u^{n_1} v^{n_2}}$ of the PDF $p(u, v)$ exist for (n_1, n_2) in a large domain of the complex plane \mathbb{C} . These are given by an analytical continuation of (29) and allow, using a Mellin-Barnes type contour integral formula, to recover the Laplace transform of $p(u, v)$ in a rigorous manner. In this paper and as in the log-Gamma case, we adapt this observation to conjecture a formula for the LT of $Z_t(x)$ by using an analytical continuation of the formula for the integer moments that we compute using the replicas Bethe ansatz.

The search of such a PDF $p(u, v)$ with moments given by (29) is in general a difficult task. Notice however that it is sufficient to examine the case $|q| \leq 1$. Indeed, using that $(\frac{1}{a}; \frac{1}{q})_n = (-a)^{-n} q^{-\frac{n(n-1)}{2}} (a; q)_n$ one easily sees on (29) that the simultaneous change $q \rightarrow 1/q$, $\mu \rightarrow 1/\mu$ and $\nu \rightarrow 1/\nu$ in $\psi_{q,\mu,\nu}(n_1, n_2)$ just multiplies it by power-law terms, easily absorbed in rescaling of the variables (u, v) and which cannot break the integrability of the model.

III. INTEGRABLE POLYMER MODELS

A. The $|q| < 1$ case.

Without loss of generality we restrict ourselves in the following to $|q| < 1$, and we further restrict to $q, \mu, \nu \in \mathbb{R}$.

We now consider the case where the moments exist at least up to the second moments (i.e. $n_{\max} \geq 2$). Let us consider the random variable $z_x = u + xv$, $x \in \mathbb{R}$. A simple calculation from (29) gives that its variance is:

$$\overline{z_x^2} - \overline{z_x}^2 = \frac{(\mu-1)(1-q)(\mu-\nu)(\mu x-1)(\mu x-\nu)}{\mu^2(\nu-1)^2(\nu q-1)} \quad (30)$$

Under our assumptions this expression must be positive. Since the polynomial in x changes sign at $x = \frac{\nu}{\mu}$ and $x = \frac{1}{\mu}$ this clearly rules out the generic case for $q < 1, \mu, \nu$.

We must thus look for degenerations with $\nu/\mu = 1/\mu$ so that the variance of z_x can eventually be positive $\forall x$. The various cases are studied systematically in Appendix A where we show that the only possibility for the existence of such a PDF is in the $q \rightarrow 1$ limit which we now study in details. Moreover, we also show there that the limit $q \rightarrow 1$ and $\mu, \nu \rightarrow 1$ at the same speed than q , contains all the interesting cases.

Finally, note that the above considerations do not rule out completely the existence of an integrable polymer model with $q < 1$ since it could correspond to a model with $n_{\max} < 2$. From the discussion of the previous Section, this would involve however an exhaustive study of the possible analytical continuations of (29) which goes beyond the present work.

B. The $q \rightarrow 1$ limit

1. Form of the moments.

Let us now discuss the $q \rightarrow 1$ limit. We use that at fixed n, a , $(q^a; q)_n \simeq_{q \rightarrow 1} (1-q)^n (a)_n$, where $(a)_n = a(a+1)\dots(a+n-1)$ is the standard Pochhammer symbol. This is easily seen setting $q = e^{-\epsilon}$ and taking $\epsilon \rightarrow 0$. The ratio $\frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}$ thus tends to the standard binomial coefficient C_n^m .

To obtain a meaningful limit we scale $\nu = q^{\alpha+\beta}$, $\mu = q^\beta$. In this case, one gets as $q \rightarrow 1$:

$$\overline{u^{n_1} v^{n_2}} = (\epsilon_1)^{n_1} (\epsilon_2)^{n_2} \frac{(\alpha)_{n_1} (\beta)_{n_2}}{(\alpha + \beta)_{n_1 + n_2}}, \quad (31)$$

where we have added two power law terms with $(\epsilon_1, \epsilon_2) \in \{-1, 1\}^2$. We were allowed to do it if we start to examine the moment problem for real variables. These two additional parameters are then tuned so that (u, v) are positive random variables. Since they are a-priori arbitrary we must examine all cases. Other interesting limits can also be considered but they can all be obtained from (31) as a new limit (see Appendix A). Note that (31) implies, $\forall n \in \mathbb{N}$,

$$\overline{(u/\epsilon_1 + v/\epsilon_2)^n} = \sum_{n_1=0}^n C_n^{n_1} \frac{(\alpha)_{n_1} (\beta)_{n-n_1}}{(\alpha + \beta)_n} = 1, \quad (32)$$

so that, except maybe in some marginal cases discussed in Appendix B, this implies that u and v are correlated as $\epsilon_2 u + \epsilon_1 v = \epsilon_1 \epsilon_2$. In Appendix B, we initiate a more systematic study of all possible cases as ϵ_i are varied.

2. The Beta Polymer and the Strict-Weak limit

The case of $(\epsilon_1, \epsilon_2) = (1, 1)$, $\alpha > 0$ and $\beta > 0$ indeed corresponds to the moments of two positive random variables. In this case, one has $v = 1 - u$ and $u \in [0, 1]$ is distributed according to Beta random variable:

$$u \sim \text{Beta}(\alpha, \beta) \iff p_{\alpha, \beta}(u) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1}$$

$$v = 1 - u \quad , \quad \overline{u^{n_1} v^{n_2}} = \frac{(\alpha)_{n_1} (\beta)_{n_2}}{(\alpha + \beta)_{n_1 + n_2}}. \quad (33)$$

Where from now on \sim means distributed as or the equivalence in probability. Note that Beta distributions satisfy $\text{Beta}(\alpha, \beta) \sim 1 - \text{Beta}(\beta, \alpha)$, and that in the Beta polymer model, interverting horizontal and vertical edges amounts to permute α and β .

Note that for this distribution of (u, v) , the formula for the moments (33) can be extended to the complex moments and admits a more general expression as

$$\overline{u^{s_1} v^{s_2}} = \frac{\Gamma(\alpha + \beta) \Gamma(\alpha + s_1) \Gamma(\beta + s_2)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha + \beta + s_1 + s_2)} \quad (34)$$

which is valid for arbitrary complex numbers $(s_1, s_2) \in \mathbb{C}^2$ in the domain $Re(s_1) > -\alpha$ and $Re(s_2) > -\beta$. The corresponding Directed Polymer model was introduced and studied in⁵⁹. As already observed there, this model also contains the Strict-Weak polymer model introduced in⁵⁶ as a limit $\beta \rightarrow \infty$:

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \overline{(\beta u)^{n_1} v^{n_2}} &= (\alpha)_{n_2} = \frac{\Gamma(\alpha + n_1)}{\Gamma(\alpha)} \\ (\beta u, v) &\sim (\beta \text{Beta}(\alpha, \beta), (1 - \text{Beta}(\alpha, \beta))) \sim_{\beta \rightarrow \infty} (\text{Gamma}(\alpha), 1) , \end{aligned} \quad (35)$$

which corresponds to a Strict-Weak polymer model with random Boltzmann weights on *vertical edges* distributed according to a Gamma distribution of parameter $\alpha > 0$, more precisely the rescaled Boltzmann weight $u' = \beta u$ is distributed according to a PDF $p_\alpha(u')$ such that

$$u' \sim \text{Gamma}(\alpha) \iff p_\alpha(u') = \frac{1}{\Gamma(\alpha)} (u')^{-1+\alpha} e^{-u'} . \quad (36)$$

A second, and completely symmetric, Strict-Weak DP limit exists for $\alpha \rightarrow \infty$ at fixed β , with random $\text{Gamma}(\beta)$ weights on horizontal edges.

3. The Inverse-Beta polymer

We now investigate the case $(\epsilon_1, \epsilon_2) = (1, -1)$ with $\beta > 0$ and $\alpha + \beta < 1$, and for convenience let us introduce the parameter γ as:

$$\gamma := 1 - (\alpha + \beta) \quad (37)$$

In this case, a solution to the moment problem (31) is given by $v = u - 1$ (in agreement with the general argument proposed above) and $u \in [1, +\infty[$ distributed as

$$\begin{aligned} \tilde{p}_{\gamma, \beta}(u) &= \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} , \quad v = u - 1 , \quad u \in [1, +\infty[, \gamma > 0 \\ \overline{u^{n_1} v^{n_2}} &= (-1)^{n_2} \frac{(\alpha)_{n_1} (\beta)_{n_2}}{(\alpha + \beta)_{n_1 + n_2}} \text{ for } n_1 \leq 1 - \alpha = \gamma + \beta , \quad n_1 + n_2 \leq 1 - (\alpha + \beta) = \gamma . \end{aligned} \quad (38)$$

In this case the moments problem (31) is indeed truly solved only for $n_1 + n_2 \leq \gamma$ since the moments cease to exist beyond this bound, due to divergence for large values of u, v . However there is a more general expression of the moments for complex $(s_1, s_2) \in \mathbb{C}^2$ with $Re(s_1 + s_2) \leq \gamma$ and $Re(s_2) > -\beta$

$$\overline{u^{s_1} v^{s_2}} = \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \frac{\Gamma(\gamma - s_1 - s_2) \Gamma(\beta + s_2)}{\Gamma(\gamma + \beta - s_1)} . \quad (39)$$

Using the analytical continuation of the Gamma function to the full complex plane, one thus see, using (39), that the moment problem (31) is indeed solved in an analytical continuation sense. This situation is very similar to the case of the log-Gamma polymer which the present model generalizes, as we show below. Note that for the present model, the variable $1/u$ is distributed according to a Beta distribution of parameters γ and β , $1/u \sim \text{Beta}(\gamma, \beta)$ and for this reason we call this model *the Inverse-Beta polymer*. This observation renders the proof of the convergence of this model to the log-Gamma polymer immediate. Indeed, one has

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \overline{\left(\frac{u}{\beta}\right)^{n_1} \left(\frac{v}{\beta}\right)^{n_2}} &= \frac{\Gamma(\gamma - (n_1 + n_2))}{\Gamma(\gamma)} \\ \left(\frac{u}{\beta}, \frac{v}{\beta}\right) &\sim \left(\frac{1}{\beta \text{Beta}(\gamma, \beta)}, \frac{1 - \text{Beta}(\gamma, \beta)}{\beta \text{Beta}(\gamma, \beta)}\right) \sim_{\beta \rightarrow \infty} \frac{(1, 1)}{\text{Gamma}(\gamma)} . \end{aligned} \quad (40)$$

And this limit thus corresponds to a model of polymer with on sites Boltzmann weights (since the weights on neighboring links are equals in the limit) distributed according to an inverse Gamma distribution, i.e. the log-Gamma polymer. This analysis thus unveil a natural duality between known integrable directed polymer models, as can be

seen comparing (35) and (40). However, more surprisingly, this model also contains the Strict-Weak polymer model as a limit. Indeed, one has

$$\lim_{\gamma \rightarrow \infty} \overline{u^{n_1}(\gamma v)^{n_2}} = \frac{\Gamma(\beta + n_2)}{\Gamma(\beta)}$$

$$(u, \gamma v) \sim \left(\frac{1}{\text{Beta}(\gamma, \beta)}, \frac{\gamma(1 - \text{Beta}(\gamma, \beta))}{\text{Beta}(\gamma, \beta)} \right) \sim_{\gamma \rightarrow \infty} (1, \text{Gamma}(\beta)) \quad (41)$$

which corresponds to a Strict-Weak polymer model with Boltzmann weights on *horizontal edges* distributed with a Gamma distribution of parameter $\beta > 0$. In terms of the partition sum, the convergence of the Inverse-Beta model to the log-Gamma and Strict-Weak (2) is easily obtained using (40) and (41). Note that one can formally take a limit on the moments of the Beta polymer to obtain the moments of the log-Gamma polymer. Indeed, taking on the moments appearing in (33) $\alpha + \beta = 1 - \gamma$ fixed and letting $\alpha \rightarrow \infty$, one obtain

$$\lim_{|\alpha|, |\beta| \rightarrow \infty, \alpha + \beta = 1 - \gamma} \left(\frac{u}{-\alpha} \right)^{n_1} \left(\frac{v}{-\beta} \right)^{n_2} = \frac{(-1)^{n_1 + n_2}}{(1 - \gamma)^{n_1 + n_2}} = \frac{\Gamma(\gamma - (n_1 + n_2))}{\Gamma(\gamma)}. \quad (42)$$

But in doing so, the parameters α and β passes through region where the PDF $\text{Beta}(\alpha, \beta)$ is not normalizable and the convergence of the Beta to the log-Gamma polymer thus does not hold in probability. The situation and relations between this different polymer models is summarized in Fig. 2. Notice that parts of this scheme remain empty, and there still remains some room for new integrable models with moments of the form (31). In Appendix B we attempt a first step in this direction by studying different analytical continuations of (31),

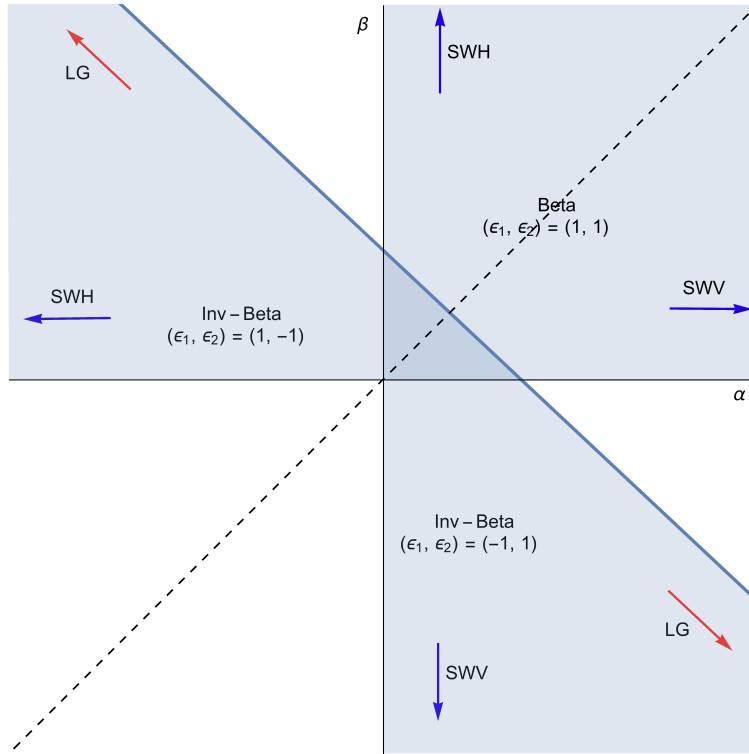


FIG. 2: Duality between polymers models in the (α, β) plane. The dashed line represents the axe of symmetry $\alpha \leftrightarrow \beta$, or equivalently the symmetry between vertical and horizontal edges. The blue line indicates the line $\alpha + \beta = 1$ or equivalently $\gamma = 1 - (\alpha + \beta) = 0$. Limiting polymer models are indicated by red arrows for the log-Gamma (LG) and blue arrows for the Strict-Weak (with weights either on horizontal edges (SWH) or vertical edges (SWV)). We also emphasize the values of (ϵ_1, ϵ_2) which corresponds to the polymer considered. Notice that the region $\alpha > 0, \beta > 0$ and $\gamma < 1$ is a region of coexistence of the Inverse-Beta and the Beta polymer, only distinguished by the value of (ϵ_1, ϵ_2) .

IV. STUDY OF THE INVERSE-BETA POLYMER

We now turn to the analysis of the Inverse-Beta polymer. In Section IV A we use the Bethe ansatz solvability of the model to obtain formulas for the firsts (i.e. those that exist) integer moments of the partition sum of the model. In Section IV B we use the prescription already used in⁵⁵ for the log-Gamma polymer to conjecture a formula for the Laplace transform of the partition sum from the knowledge of its moments. Based on this conjecture, we show in Section IV C the KPZ universality of the model. Finally, we study in Section IV D a zero temperature model associated to the Inverse-Beta polymer.

A. Moments Formula and Coordinate Bethe Ansatz

1. Coordinate Bethe Ansatz

The moments of the Boltzmann weights of the Inverse-Beta polymer read (in the following we keep the notations and coordinates introduced in the general setting of the precedent section)

$$\overline{u^{n_1} v^{n_2}} = (-1)^{n_2} \frac{(\alpha)_{n_1} (\beta)_{n_2}}{(\alpha + \beta)_{n_1 + n_2}} . \quad (43)$$

where $\alpha < 0$, $\beta > 0$ and $\alpha + \beta < 1$. As we showed in Section III B, this model is integrable using a coordinate Bethe ansatz Eq. (14) with a two body S -matrix $S(z_i, z_j)$ given by (19). Its parameters are calculated from the second moment equation Eq. (18) and their definition (43), leading to:

$$\mathbf{a} = \mathbf{c} = -\frac{1}{1 + \alpha + \beta} = \frac{1}{\gamma - 2} \quad , \quad \mathbf{b} = \frac{-1 + \alpha + \beta}{1 + \alpha + \beta} = \frac{\gamma}{\gamma - 2} \quad , \quad \gamma = 1 - (\alpha + \beta) \quad (44)$$

where we have recalled the definition of the parameter γ . We now introduce

$$\bar{c} = \frac{4}{\gamma - 1} = -\frac{4}{\alpha + \beta} \\ z_j = e^{i\lambda_j} \quad , \quad t_j = i \tan\left(\frac{\lambda_j}{2}\right) = \frac{z_j - 1}{z_j + 1} \quad , \quad z_j = \frac{1 + t_j}{1 - t_j} . \quad (45)$$

In the following we suppose $\bar{c} > 0$, i.e. $\gamma > 1$. As in the log-Gamma case, this is only a technical assumption that allows us to use the coordinate Bethe ansatz to compute the $n < \gamma$ first moments of the partition sum, and we will specify when the validity of some results extends to $\gamma < 1$. Using these notations, it is a simple exercise to check that the S -matrix of the Inverse-Beta polymer can be expressed as

$$S(z_i, z_j) = \frac{2t_j - 2t_i + \bar{c}}{2t_j - 2t_i - \bar{c}} . \quad (46)$$

Remarkably, it is equal to the S -matrix of the log-Gamma polymer studied in our previous work⁵⁵. Hence, the Bethe eigenfunctions of this model can be taken as the one already introduced for the log-Gamma polymer, namely

$$\tilde{\psi}_\mu(x_1, \dots, x_n) = \sum_{\sigma \in S_n} A_\sigma \prod_{\alpha=1}^n z_{\sigma(\alpha)}^{x_\alpha} \quad , \quad A_\sigma = \prod_{1 \leq \alpha < \beta \leq n} \left(1 + \frac{\bar{c}}{2(t_{\sigma(\alpha)} - t_{\sigma(\beta)})}\right) . \quad (47)$$

Note that it only differs from the solution (20) proposed in²⁴ by a global multiplicative constant. Following the same approach than in⁵⁵, we now study the model using periodic boundary conditions and look for eigenstates of the transfer matrix such that $\psi_\mu(x_1, \dots, x_j + L, \dots, x_n) = \psi(x_1, \dots, x_n)$. This imposes the Bethe equations

$$e^{i\lambda_i L} = \prod_{1 \leq j \leq n, j \neq i} \frac{2t_i - 2t_j + \bar{c}}{2t_i - 2t_j - \bar{c}} \quad , \quad i = 1, \dots, n \quad (48)$$

Note that this is only a convenient choice and should have no effects on $Z_t(x)$ as long as $t < L$ as discussed there. We will now recall some useful properties of the eigenstates (47) that were obtained⁵⁵ and generalize some of them.

2. Recall of some properties of the eigenstates

a. A weighted scalar product The eigenfunctions (47) form a basis of the set of periodic functions of n variables on \mathbb{N}^n . They are orthogonal with respect to the following scalar product

$$\langle \phi, \psi \rangle = \sum_{(x_1, \dots, x_n) \in \{0, \dots, L-1\}^n} \frac{1}{\prod_x h_{\sum_{\alpha=1}^n \delta_{x, x_\alpha}}} \phi^*(x_1, \dots, x_n) \psi(x_1, \dots, x_n) \quad , \quad h_n = \prod_{k=0}^{n-1} \frac{4}{4 - k\bar{c}} = (\gamma - 1)^n \frac{\Gamma(\gamma - n)}{\Gamma(\gamma)} . \quad (49)$$

b. The string solution In the large L limit, the solutions of the Bethe equations (48) organized themselves into strings. Each set $\{t_\alpha\}$ that solve (48) is given by partitioning n into n_s strings, each string containing m_j particles where the index $j = 1, \dots, n_s$ labels the string. Inside a string, the t_α are given by (we use the notations of⁵⁵):

$$t_\alpha = t_{j,a} = i \frac{k_j}{2} + \frac{\bar{c}}{4} (m_j + 1 - 2a) + \frac{\delta_{j,a}}{2} \quad (50)$$

where we introduced an index $a = 1, \dots, m_j$ that labels the rapidities inside a string, $\frac{k_j}{2} \in \mathbb{R}$ denotes their common imaginary part and $\delta_{j,a}$ are deviations that fall off exponentially with L . In the large L limit, the strings behave as independent free particles with total momentum $K_j = \sum_{a=1}^{m_j} \lambda_{j,a} \in [-m_j\pi, m_j\pi]$. In particular, in the large L limit, the sum over all eigenstates can be computed as

$$\sum_{m_j \text{ string-states}} \rightarrow \frac{L}{2\pi} \int_{-m_j\pi}^{m_j\pi} dK_j \rightarrow \frac{L}{2\pi} \int_{-\infty}^{\infty} dk_j \sum_{a=1}^{m_j} \frac{1}{1 - t_{j,a}^2} . \quad (51)$$

We will also need the norm of an eigenstate composed of strings in the large L limit. This was computed in⁵⁵.

$$\|\mu\|^2 = n! L^{n_s} \prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + \bar{c}^2(m_i + m_j)^2}{4(k_i - k_j)^2 + \bar{c}^2(m_i - m_j)^2} \prod_{j=1}^{n_s} \left[\frac{m_j}{\bar{c}^{m_j-1}} \left(\sum_{a=1}^{m_j} \frac{1}{1 - t_{j,a}^2} \right) \prod_{b=1}^{m_j} (1 - t_{j,b}^2) \right] \quad (52)$$

c. Energy-momentum of the strings Although the eigenfunctions are the same as the one for the log-Gamma polymer, the *eigenvalues* are different. The eigenvalue of the transfer matrix T_n associated to an eigenstate ψ_μ was given in (15) as $\Lambda_\mu = \prod_{i=1}^n (\bar{u} + \bar{v} z_i^{-1})$ and depends only on the first moments of the weights. Inserting their values from (43) and taking into account that for a string state, it is a product of string contributions, we obtain $\Lambda_\mu = \prod_{j=1}^{n_s} \Lambda_j$ with:

$$\begin{aligned} \Lambda_j &= \prod_{a=1}^{m_j} \left(\frac{\alpha}{\alpha + \beta} - \frac{\beta}{\alpha + \beta} \frac{1 - t_{j,a}}{1 + t_{j,a}} \right) = \frac{\left(-\beta - \frac{ik_j}{\bar{c}} - \frac{\gamma}{2} - \frac{m_j}{2} + 1 \right)_{m_j}}{\left(-\frac{ik_j}{\bar{c}} - \frac{\gamma}{2} - \frac{m_j}{2} + 1 \right)_{m_j}} \\ &= \frac{\Gamma\left(\beta + \frac{ik_j}{\bar{c}} + \frac{\gamma}{2} + \frac{m_j}{2}\right) \Gamma\left(\frac{ik_j}{\bar{c}} + \frac{\gamma}{2} - \frac{m_j}{2}\right)}{\Gamma\left(\beta + \frac{ik_j}{\bar{c}} + \frac{\gamma}{2} - \frac{m_j}{2}\right) \Gamma\left(\frac{ik_j}{\bar{c}} + \frac{\gamma}{2} + \frac{m_j}{2}\right)} . \end{aligned} \quad (53)$$

Where, in the second line, we have rewritten the Pochhammer symbols using Gamma functions, an identity valid for integer m_j .

Another important quantity is the eigenvalue associated to the action of the unit translation operator on a string state, defined as

$$\prod_{a=1}^{m_j} z_{j,a} = \prod_{a=1}^{m_j} \frac{1 + t_{j,a}}{1 - t_{j,a}} = \frac{\Gamma(-\frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{\bar{c}}) \Gamma(\frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{\bar{c}})}{\Gamma(\frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{\bar{c}}) \Gamma(-\frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{\bar{c}})} , \quad (54)$$

an expression identical to the one obtained in⁵⁵. Finally, we will also need

$$\left(\prod_{a=1}^{m_j} \frac{1}{1 - t_{j,a}^2} \right) = \left(\frac{2}{\bar{c}} \right)^{2m_j} \left(\frac{\Gamma(-\frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{\bar{c}}) \Gamma(-\frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{\bar{c}})}{\Gamma(\frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{\bar{c}}) \Gamma(\frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{\bar{c}})} \right) . \quad (55)$$

3. Moments formula

We have now all the ingredients to compute the integer moments of the partition sum. As will appear clearly in the following, it is convenient to start with an initial condition

$$Z_{t=0}(x) = w_{0,0} \delta_{x,0} , \quad (56)$$

where $w_{0,0}$ is Boltzmann weight, statistically independent of the others, and distributed with an inverse Gamma distribution of parameter γ . The problem with initial condition $Z_{t=0}(x) = \delta_{x,0}$ is obviously simply connected to this one and the details of the relations are given in Appendix C. In terms of the wave-function $\psi_t(x_1, \dots, x_n)$, the initial conditions reads $\psi_{t=0}(x_1, \dots, x_n) = \frac{\Gamma(\gamma-n)}{\Gamma(\gamma)} \prod_{i=1}^n \delta_{x_i,0}$ and we use the scalar product (49) to decompose it on the Bethe eigenstates. Using this decomposition we obtain

$$\psi_t(x_1, \dots, x_n) = \sum_{\mu} \frac{\Gamma(\gamma-n)n!}{\Gamma(\gamma)h_n \|\psi_{\mu}\|^2} (\Lambda_{\mu})^t \psi_{\mu}(x_1, \dots, x_n) . \quad (57)$$

In particular,

$$\overline{Z_t(x)^n} = \sum_{\mu} \frac{\Gamma(\gamma-n)(n!)^2}{\Gamma(\gamma)h_n \|\psi_{\mu}\|^2} (\Lambda_{\mu})^t \left(\prod_{\alpha=1}^n z_{\alpha} \right)^x . \quad (58)$$

Replacing in this expression each terms by its value in the large L limit, one obtains:

$$\begin{aligned} \overline{Z_t(x)^n} = & \frac{\Gamma(\gamma-n)(n!)^2}{\Gamma(\gamma)h_n} \sum_{n_s=1}^n \frac{1}{n_s!} \sum_{(m_1, \dots, m_{n_s})_n} \prod_{j=1}^{n_s} \int_{-\infty}^{+\infty} \left[\frac{dk_j}{2\pi} \sum_{a=1}^{m_j} \frac{1}{1-t_{j,a}^2} \right] \frac{1}{n!} \prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + \bar{c}^2(m_i - m_j)^2}{4(k_i - k_j)^2 + \bar{c}^2(m_i + m_j)^2} \\ & \prod_{j=1}^{n_s} (\bar{c})^{m_j-1} \frac{1}{m_j (\sum_{a=1}^{m_j} \frac{1}{1-t_{j,a}^2}) \prod_{b=1}^{m_j} (1-t_{j,b}^2)} (\Lambda_j)^t \prod_{b=1}^{m_j} \left(\frac{1+t_{j,b}}{1-t_{j,b}} \right)^x . \end{aligned} \quad (59)$$

Where we have written the sum over all eigenstates as $\sum_{\mu} = \sum_{n_s=1}^n \frac{1}{n_s!} \sum_{(m_1, \dots, m_{n_s})_n} \sum_{m_j \text{ string-states}}$, where $\sum_{(m_1, \dots, m_{n_s})_n}$ means summing over all n_s -uplets (m_1, \dots, m_{n_s}) such that $\sum_{i=1}^{n_s} m_i = n$, and the $n_s!$ factor avoids multiple counting of a same string state. Rearranging this formula and rescaling $k \rightarrow \bar{c}k$, we finally obtain:

$$\begin{aligned} \overline{Z_t(x)^n} = & n! \sum_{n_s=1}^n \frac{1}{n_s!} \sum_{(m_1, \dots, m_{n_s})_n} \prod_{j=1}^{n_s} \int_{-\infty}^{+\infty} \frac{dk_j}{2\pi} \prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2} \\ & \prod_{j=1}^{n_s} \frac{1}{m_j} \left(\frac{\Gamma(-\frac{m_j}{2} + \frac{\gamma}{2} - ik_j)}{\Gamma(\frac{m_j}{2} + \frac{\gamma}{2} - ik_j)} \right)^{1+x} \left(\frac{\Gamma(-\frac{m_j}{2} + \frac{\gamma}{2} + ik_j)}{\Gamma(\frac{m_j}{2} + \frac{\gamma}{2} + ik_j)} \right)^{1-x+t} \left(\frac{\Gamma(\beta + ik_j + \frac{\gamma}{2} + \frac{m_j}{2})}{\Gamma(\beta + ik_j + \frac{\gamma}{2} - \frac{m_j}{2})} \right)^t , \end{aligned} \quad (60)$$

valid for $n < \gamma$. The convergence of the various integrals is algebraic, the integrand being $O(1/k_j^{2m_j})$ as can be checked by rewriting the quotient of Gamma functions as Pochhammer symbols. This formula was checked using direct numerical integrations for low values of $t \leq 2$, $x \leq 2$ and $n \leq 2$. The case $t = 0$ is already non-trivial since it confirms the completeness of the eigenstates. Note that if one chooses the initial condition $Z_{t=0}(x) = \delta_{x,0}$, then $\overline{Z_t(x)^n}$ is trivially given by (60) with an additional factor of $\Gamma(\gamma)/\Gamma(\gamma-n)$ in front.

Degenerations towards the log-Gamma and Strict-Weak polymers: Since the Inverse-Beta polymer contains the log-Gamma polymer and the Strict-Weak polymer as limits (see (40) and (41)), (60) also contains moments formula for the Strict-Weak and log-Gamma cases as we show now.

- The moments of the log-Gamma polymer are obtained as the limit $\overline{(Z_t^{LG}(x))^n} = \lim_{\beta \rightarrow \infty} \frac{1}{\beta^{nt}} \overline{Z_t(x)^n}$, where $Z_t^{LG}(x)$ is the partition sum of the log-Gamma polymer. Indeed, the factor $\frac{1}{\beta^{nt}}$ exactly cancels the divergence of the last quotient of Gamma functions in (60), leading to the formula (54) of⁵⁵. Let us recall that the present coordinates are $t = T$ and $x = t/2 + X$ as a function of those, T, X (but denoted there t, x) of that work.
- We now obtain a moment formula for the Strict-Weak polymer with initial condition $Z_t^{SW}(x=0) = \delta_{x,0}$, following (41), we consider the limit $\overline{(Z_t^{SW}(x))^n} = \lim_{\gamma \rightarrow \infty} \frac{\Gamma(\gamma)}{\Gamma(\gamma-n)} \gamma^{nx} \overline{Z_t(x)^n}$. In this case, the point-wise limit

of the integrand cannot be taken as simply and we need to first perform the change of variables $k_j \rightarrow k_j + i\frac{\gamma}{2}$. We obtain

$$\begin{aligned} \overline{(Z_t^{SW}(x))^n} &= \lim_{\gamma \rightarrow \infty} \frac{\Gamma(\gamma)}{\Gamma(\gamma - n)} \gamma^{nx} n! \sum_{n_s=1}^n \frac{1}{n_s!} \sum_{(m_1, \dots, m_{n_s})_n} \prod_{j=1}^{n_s} \int_{L^n} \frac{dk_j}{2\pi} \prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2} \\ &\prod_{j=1}^{n_s} \frac{1}{m_j} \left(\frac{\Gamma(-\frac{m_j}{2} + \gamma - ik_j)}{\Gamma(\frac{m_j}{2} + \gamma - ik_j)} \right)^{1+x} \left(\frac{\Gamma(-\frac{m_j}{2} + ik_j)}{\Gamma(\frac{m_j}{2} + ik_j)} \right)^{1-x+t} \left(\frac{\Gamma(\beta + ik_j + \frac{m_j}{2})}{\Gamma(\beta + ik_j - \frac{m_j}{2})} \right)^t. \end{aligned} \quad (61)$$

Where $L = -i\frac{\gamma}{2} + \mathbb{R}$. Since the integral over k_j quickly converges as $O(1/k_j^{2m_j})$, we can now close the different contours of integrations on the upper half plane before taking the limit $\gamma \rightarrow \infty$. This leads to:

$$\begin{aligned} \overline{(Z_t^{SW}(x))^n} &= n! \sum_{n_s=1}^n \frac{1}{n_s!} \sum_{(m_1, \dots, m_{n_s})_n} \prod_{j=1}^{n_s} \int_{\tilde{L}^n} \frac{dk_j}{2\pi} \prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2} \\ &\prod_{j=1}^{n_s} \frac{1}{m_j} \left(\frac{\Gamma(-\frac{m_j}{2} + ik_j)}{\Gamma(\frac{m_j}{2} + ik_j)} \right)^{1-x+t} \left(\frac{\Gamma(\beta + ik_j + \frac{m_j}{2})}{\Gamma(\beta + ik_j - \frac{m_j}{2})} \right)^t, \end{aligned} \quad (62)$$

where \tilde{L} is an horizontal line that stays below all the poles of the integrand. This formula is formal because the resulting integral does not converge, but one must remember that we have formally already closed the contours of integrations. Computing the integral on k_i thus just amounts at taking the sum over the residues of all the poles of the integrands except those of the type $k_i = k_j - iA$ where $A > 0$ (since the contours have been closed on the upper half-plane).

B. Fredholm determinant formulas and KPZ universality

In this section, we use the formula (60) to obtain the Laplace transform of the distribution of $Z_t(x)$,

$$g_{t,x}(u) = \overline{\exp(-uZ_t(x))}. \quad (63)$$

The issue of obtaining this generating function from the sole knowledge of the integer moments of the partition sum was thoroughly discussed in⁵⁵ and here we follow the same route.

1. The moment generating function

We start by computing the moment generating function

$$g_{t,x}^{mom}(u) = \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \overline{Z_t(x)^n}. \quad (64)$$

where $u > 0$. Here, though $\overline{Z_t(x)^n}$ is only defined for $n \leq \gamma$, the right hand side of formula (60) is well defined for $n \in \mathbb{N}$ (except if $\gamma \in \mathbb{N}$) and we take advantage of this analytical continuation to perform the sum (64). Note that this object has no reason to correspond to the Laplace transform of $Z_t(x)$ but⁶⁷, as in the log-Gamma case, we will use it to conjecture a formula for the true Laplace transform $g_{t,x}(u)$ defined in (63). Since we perform the sum over $n \in \mathbb{N}$, the constrained sum appearing in (60) becomes free summation and one can write

$$g_{t,x}^{mom}(u) = 1 + \sum_{n_s=1}^{+\infty} \frac{1}{n_s!} Z(n_s, u) \quad (65)$$

⁶⁷ And indeed it is not, a simple reason being that, just as the Laplace transform of the PDF $\tilde{p}_{\gamma,\beta}$ of the Boltzmann weights of the Inverse-Beta polymer (see (38)), the Laplace transform of $Z_t(x)$ is not an analytic function. See also Appendix D for more details on this question.

where

$$Z(n_s, u) = \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{dk_j}{2\pi} \prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2} \prod_{j=1}^{n_s} \frac{(-u)^{m_j}}{m_j} \left(\frac{\Gamma(-\frac{m_j}{2} + \frac{\gamma}{2} - ik_j)}{\Gamma(\frac{m_j}{2} + \frac{\gamma}{2} - ik_j)} \right)^{1+x} \left(\frac{\Gamma(-\frac{m_j}{2} + \frac{\gamma}{2} + ik_j)}{\Gamma(\frac{m_j}{2} + \frac{\gamma}{2} + ik_j)} \right)^{1-x+t} \left(\frac{\Gamma(\beta + ik_j + \frac{\gamma}{2} + \frac{m_j}{2})}{\Gamma(\beta + ik_j + \frac{\gamma}{2} - \frac{m_j}{2})} \right)^t \quad (66)$$

In this formula and following⁵⁵, one recognizes the structure of a Fredholm determinant

$$g_{t,x}^{mom}(u) = \text{Det} (I + K_{t,x}^{mom}) \quad (67)$$

with the kernel:

$$K_{t,x}^{mom}(v_1, v_2) = \sum_{m=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dk}{\pi} (-u)^m e^{-2ik(v_1 - v_2) - m(v_1 + v_2)} \left(\frac{\Gamma(-\frac{m}{2} + \frac{\gamma}{2} - ik)}{\Gamma(\frac{m}{2} + \frac{\gamma}{2} - ik)} \right)^{1+x} \left(\frac{\Gamma(-\frac{m}{2} + \frac{\gamma}{2} + ik)}{\Gamma(\frac{m}{2} + \frac{\gamma}{2} + ik)} \right)^{1-x+t} \left(\frac{\Gamma(\beta + ik + \frac{\gamma}{2} + \frac{m}{2})}{\Gamma(\beta + ik + \frac{\gamma}{2} - \frac{m}{2})} \right)^t \quad (68)$$

and $K_{t,x}^{mom} : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$, so that the two auxiliary integration variables v_1 and v_2 are positive⁶⁸.

2. The Laplace transform as a Fredholm determinant

We now use the same prescription used in⁵⁵ to obtain a conjecture for the Laplace transform $g_{t,x}(u)$ from the moment generating function $g_{t,x}^{mom}(u)$. It consists in rewriting the sum over m in the Kernel $K_{t,x}^{mom}$ as a Mellin-Barnes integral. In Appendix D we also show how this type of manipulation efficiently works on a simpler object, namely the Laplace transform of the PDF $\tilde{p}_{\gamma,\beta}$ defined in (38). We thus conjecture, $g_{tx}(u) = \text{Det} (I + K_{tx})$ with

$$K_{t,x}(v_1, v_2) = \int_{-\infty}^{+\infty} \frac{dk-1}{\pi} \frac{1}{2i} \int_C \frac{ds}{\sin(\pi s)} u^s e^{-2ik(v_1 - v_2) - s(v_1 + v_2)} \left(\frac{\Gamma(-\frac{s}{2} + \frac{\gamma}{2} - ik)}{\Gamma(\frac{s}{2} + \frac{\gamma}{2} - ik)} \right)^{1+x} \left(\frac{\Gamma(-\frac{s}{2} + \frac{\gamma}{2} + ik)}{\Gamma(\frac{s}{2} + \frac{\gamma}{2} + ik)} \right)^{1-x+t} \left(\frac{\Gamma(\beta + ik + \frac{\gamma}{2} + \frac{s}{2})}{\Gamma(\beta + ik + \frac{\gamma}{2} - \frac{s}{2})} \right)^t \quad (69)$$

where $C = a + i\mathbb{R}$ with $0 < a < \min(1, \gamma)$ and $K_{t,x} : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$. As in the log-Gamma case, we expect this formula to be also valid for $0 < \gamma < 1$. Note that in going from (68) to (69) we have to choose an analytical continuation to go from $m \in \mathbb{N}$ to $s \in \mathbb{C}$. Here the chosen analytical continuation is the most natural one in the sense that it generalizes the one used for the log-Gamma polymer in⁵⁵, and also mimics the calculation of Appendix D. This Kernel is the one that is naturally obtained from the Bethe Ansatz and its structure is reminiscent of the string solution: the integral over s encodes for the contributions of the different types of strings, whereas the integral over k is the summation on the momenta of the strings. As shown in⁵⁵ (section 11), it is also possible to rewrite $g_{tx}(u)$ as the Fredholm determinant of another Kernel which contains one less integral. Since the proof is strictly analogous to the case of the log-Gamma polymer, we only give here the final result: we also have $g_{tx}(u) = \text{Det} (I + K_{tx}^{BA})$ where

$$K_{t,x}^{BA}(z, z') = \int_{2a+\tilde{a}+i\mathbb{R}} dw \frac{1}{4\pi(w - z')} \frac{1}{\sin(\pi(w - z))} u^{w-z} \left(\frac{\Gamma(\gamma + a - w)}{\Gamma(\gamma + a - z)} \right)^{1+x} \left(\frac{\Gamma(z - a)}{\Gamma(w - a)} \right)^{1-x+t} \left(\frac{\Gamma(w - a + \beta)}{\Gamma(z - a + \beta)} \right)^t \quad (70)$$

where $K_{t,x}^{BA} : L^2(a + \tilde{a} + i\mathbb{R}) \rightarrow L^2(a + \tilde{a} + i\mathbb{R})$ $0 < a < \min(1, \gamma)$ and $0 < \tilde{a} < \gamma - a$. Note that here this formula should be valid for arbitrary x , whereas for the log-Gamma polymer the analogous formula was only valid for $2x \leq t$

⁶⁸Note that this FD structure would have been broken by the initial condition $Z_{t=0}(x) = \delta_{x=0}$. (In which case (66) contains a non factorizable term of the form $\Gamma(\gamma)/\Gamma(\gamma - \sum_{i=1}^{n_s} m_i)$).

(with a mirror formula for the other case)⁶⁹. This formula is a large contour formula and an analogous small contour formula should also exist, as in the log-Gamma polymer. Let us also mention here that, following the same procedure that led in the log-Gamma case to formula (63) and (64) of⁵⁵, it is possible to directly obtain from (69) or (70) formulas for the PDF of $\log Z_t(x)$ as differences of two Fredholm determinants.

3. The Laplace transform as a n -fold integral

In⁵⁴, a formula giving an identity between a certain class of Fredholm determinant with Kernels similar to the one in (70) and a class of n -fold contour integrals was given (Theorem 2). Though the explicit form of (70) explicitly breaks the hypothesis under which this formula was proven, an analogous formula should also exist in a more general setting. Guided by this belief, we *conjecture* the following formula for the Laplace transform:

$$\overline{e^{-uZ_t(x)}} = \frac{1}{J!} \int_{(iR)^J} \prod_{j=1}^J \frac{dw_j}{2i\pi} \prod_{j \neq k=1}^J \frac{1}{\Gamma(w_j - w_k)} \left(\prod_{j=1}^J u^{w_j - a} \Gamma[a - w_j]^J \left(\frac{\Gamma(\gamma + a - w_j)}{\Gamma(\gamma)} \right)^I \left(\frac{\Gamma(w_j - a + \beta)}{\Gamma(\beta)} \right)^{I+J-2} \right), \quad (71)$$

with $0 < a < \min(1, \gamma)$, valid for $\text{Re}(u) > 0$, $1 \leq J \leq I$ and where $x = I - 1$ and $t = I + J - 2$. This can be seen as a modification to our model of the formula given in⁵³ (Theorem 3.8), and also stated in⁵⁴ (Proposition 1.4), for the log-Gamma polymer. Since ours is merely a conjecture, we have tested it numerically against direct numerical computations of the Laplace transform for various u , β and γ and for $J = 1$, $I = 1, 2, 3$ and $J = 2$, $I = 2$.

4. Degeneration towards the log-Gamma polymer.

The results of the last three paragraphs for the Laplace transforms of Z are easily seen to degenerate into the usual results for the log-Gamma polymer as $\beta \rightarrow +\infty$ using that $\exp(-uZ_t^{LG}(x)) = \lim_{\beta \rightarrow \infty} \exp(-\frac{u}{\beta t} Z_t(x))$. For example, taking the limit on formula (69), this introduces a term $\exp(-st \log(\beta))$ in the Kernel that exactly cancels the divergence of the last quotient of Gamma functions, and similarly for the other formula.

C. The large length limit and the KPZ universality.

We now study the limit of polymers of large length $t \gg 1$ for polymers with fixed endpoints $(0, 0)$ and $(t, x) = (t, (1/2 + \varphi)t)$ where $\varphi \in [-1/2, 1/2]$ represents the average angle of the path measured from the diagonal of the square lattice. The large t behavior of (69) is estimated through a saddle-point analysis similar to the one in⁵⁵ to which we refer for details. We define

$$G_\varphi(y) = \left(\frac{1}{2} + \varphi\right) \log \Gamma\left(\frac{\gamma}{2} - y\right) - \left(\frac{1}{2} - \varphi\right) \log \Gamma\left(\frac{\gamma}{2} + y\right) + \log \Gamma\left(\beta + \frac{\gamma}{2} + y\right). \quad (72)$$

So that the leading behavior of the product of Gamma functions appearing in (69) is

$$(\Gamma\Gamma)^t := \exp\left(t \left(G_\varphi\left(\frac{s}{2} + ik\right) - G_\varphi\left(-\frac{s}{2} + ik\right)\right)\right). \quad (73)$$

We now look for the critical point $(s, k) = (0, -ik_\varphi)$ such that $G''_\varphi(k_\varphi)$ is 0. This defines implicitly k_φ as

$$\left(\frac{1}{2} + \varphi\right) \psi'\left(\frac{\gamma}{2} - k_\varphi\right) - \left(\frac{1}{2} - \varphi\right) \psi'\left(\frac{\gamma}{2} + k_\varphi\right) + \psi'\left(\beta + \frac{\gamma}{2} + k_\varphi\right) = 0. \quad (74)$$

⁶⁹ Convergence of the w integral is checked using that $|\Gamma(x + iy)| \simeq \sqrt{2\pi}|y|^{x-\frac{1}{2}} e^{-\frac{\pi}{2}|y|}$.

Where $\psi = \frac{\Gamma'}{\Gamma}$ is the diGamma function . Expanding (73) around this critical point, one obtain

$$(\Gamma\Gamma)^t = \exp \left(t \left(G'_\varphi(k_\varphi)s + \frac{G''_\varphi(k_\varphi)}{6} \left(\frac{s^3}{4} - 3s\tilde{k}^2 \right) + O(s^4) \right) \right) \quad (75)$$

where $\tilde{k} = k + ik_\varphi$ and s are considered to be of the same order (this is consistent with the rest of the calculation, see below). The linear term $G'_\varphi(k_\varphi)$ corresponds to an additive constant in the limiting free energy, whereas the cubic term sets the scale of the free-energy fluctuations. To pursue the asymptotic analysis, we define

$$\begin{aligned} F_t(\varphi) &= -\log Z_t(x = (1/2 + \varphi)t) = c_\varphi t + \lambda_\varphi f_t(\varphi) \\ c_\varphi &= -G'_\varphi(k_\varphi) \quad , \quad \lambda_\varphi = \left(\frac{tG''_\varphi(k_\varphi)}{8} \right)^{\frac{1}{3}} \\ \tilde{g}_{t,\varphi}(z) &= \overline{\exp(-\lambda_\varphi(z + f_t(\varphi)))} \end{aligned} \quad (76)$$

Where $F_t(\varphi)$ is the free-energy of the directed polymer and $\tilde{g}_{t,\varphi}(z)$ is a rescaled Laplace transform which has a proper $t \rightarrow \infty$ limit for fixed $z \in \mathbb{R}$. Indeed, since $g_{t,x=(1/2+\varphi)t}(u)$ can be written $\exp(-e^{\log(u)-F_t(\varphi)})$, one has the identity $\tilde{g}_{t,\varphi}(z) = g_{t,x=(1/2+\varphi)t}(u = e^{c_\varphi t - \lambda_\varphi z})$. Rescaling $s \rightarrow s/\lambda_\varphi$, $\tilde{k} \rightarrow \frac{\tilde{k}}{\lambda_\varphi}$, $v_i \rightarrow \lambda_\varphi v_i$ and inserting $u = e^{c_\varphi t - \lambda_\varphi z}$, as well as the expansion (75), into (69), one obtains $\tilde{g}_{t,\varphi}(z) = \text{Det} \left(I + \tilde{K}_{t,\varphi}(v_1, v_2) \right)^{70}$

$$\tilde{K}_{t,\varphi}(v_1, v_2) = \int_{\mathbb{R}} \frac{d\tilde{k}}{\pi} \frac{-1}{2i} \int_C \frac{ds}{\lambda_\varphi \sin(\pi \frac{s}{\lambda_\varphi})} e^{-sz - 2i\tilde{k}(v_1 - v_2) - s(v_1 + v_2) - 4\tilde{k}^2 s + \frac{s^3}{3} + O(\frac{1}{\lambda_\varphi})} \quad (77)$$

where $\tilde{K}_{t,\varphi} : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$. The large polymer length limit $\lambda_\varphi \rightarrow \infty$ can be safely taken in this last expression, leading to a kernel \tilde{K}_∞ for which there is more freedom in the choice of the integration contour C : it should only define a convergent integral and passes to the right of zero. The $t \rightarrow \infty$ limit of the rescaled generating function can thus be written as $\lim_{t \rightarrow \infty} \tilde{g}_{t,\varphi}(z) = \text{Prob}(-f < z) = \text{Det}(I + \tilde{K}_\infty)$ where $\tilde{K}_\infty : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$ is given by

$$\tilde{K}_\infty(v_1, v_2) = - \int_{\mathbb{R}} \frac{d\tilde{k}}{2\pi} \int_{\mathbb{R}_+} dy Ai(y + z + v_1 + v_2 + \tilde{k}^2) e^{-i\tilde{k}(v_1 - v_2)} \quad (78)$$

where we used the Airy trick $\int_{\mathbb{R}} dy Ai(y) e^{ys} = e^{\frac{s^3}{3}}$ valid for $\text{Re}(s) > 0$, followed by the shift $y \rightarrow y + z + v_1 + v_2 + 4\tilde{k}^2$, the identity $\int_C \frac{ds}{2i\pi s} e^{sy} = \theta(y)$, and the rescaling $\tilde{k} \rightarrow \tilde{k}/2$. As in⁵⁵, this kernel corresponds to the Tracy-Widom GUE distribution as $\det(I + \tilde{K}_\infty) = F_2(2^{-\frac{2}{3}}z)$ where $F_2(z)$ is the standard GUE Tracy-Widom cumulative distribution function. We have thus shown

$$\lim_{t \rightarrow \infty} \text{Prob} \left(\frac{\log Z_t((1/2 + \varphi)t) + tc_\varphi}{\lambda_\varphi} < 2^{\frac{2}{3}}z \right) = F_2(z) \quad (79)$$

where the (φ -dependent) constants are determined by the system of equations:

$$0 = \left(\frac{1}{2} + \varphi \right) \psi' \left(\frac{\gamma}{2} - k_\varphi \right) - \left(\frac{1}{2} - \varphi \right) \psi' \left(\frac{\gamma}{2} + k_\varphi \right) + \psi' \left(\beta + \frac{\gamma}{2} + k_\varphi \right) \quad (80)$$

$$c_\varphi = \left(\frac{1}{2} + \varphi \right) \psi \left(\frac{\gamma}{2} - k_\varphi \right) + \left(\frac{1}{2} - \varphi \right) \psi \left(\frac{\gamma}{2} + k_\varphi \right) - \psi \left(\beta + \frac{\gamma}{2} + k_\varphi \right) \quad (81)$$

$$\lambda_\varphi = \left(-\frac{t}{8} \left(\left(\frac{1}{2} + \varphi \right) \psi'' \left(\frac{\gamma}{2} - k_\varphi \right) + \left(\frac{1}{2} - \varphi \right) \psi'' \left(\frac{\gamma}{2} + k_\varphi \right) - \psi'' \left(\beta + \frac{\gamma}{2} + k_\varphi \right) \right) \right)^{\frac{1}{3}}. \quad (82)$$

Angle of maximal probability The free energy per unit length c_φ is maximal in the direction defined by the angle φ^* such that $\frac{\partial}{\partial \varphi} c_\varphi|_{\varphi=\varphi^*} = 0$. It is easily seen from (80) that it is realized for $k_\varphi = 0$, and φ^* is thus given by

$$\varphi^* = -\frac{1}{2} \frac{\psi'(\beta + \gamma/2)}{\psi'(\gamma/2)} < 0, \quad (83)$$

⁷⁰ The extra factor $e^{-2k_\varphi \lambda_\varphi (v_1 - v_2)}$ originating from the change of variable has been removed since it is immaterial in the calculation of the Fredholm determinant.

and the optimal energy per unit length is thus

$$c^* = c_{\varphi^*} = \psi(\gamma/2) - \psi(\beta + \gamma/2) . \quad (84)$$

The amplitude of the fluctuations in the direction φ^* are

$$\begin{aligned} \lambda_{\varphi^*} &= \left(\frac{t}{8} (\psi''(\beta + \gamma/2) - \psi''(\gamma/2)) \right)^{1/3} \\ &\simeq_{\beta \rightarrow 0} \left(\frac{t}{8} \psi'''(\gamma/2) \beta \right) \\ &\simeq_{\beta \rightarrow \infty} \left(-\frac{t}{8} \psi''(\gamma/2) \right) . \end{aligned} \quad (85)$$

And one recognizes the usual log-Gamma result for $\varphi = 0$. In the log-Gamma limit $\beta \rightarrow \infty$, one recovers $\varphi^* = 0$, but the parameter $\beta > 0$ biases the DP towards the vertical direction. More precisely,

$$\begin{aligned} \varphi^* &\simeq_{\beta \rightarrow 0} -\frac{1}{2} - \frac{\psi''(\gamma/2)}{\psi'(\gamma/2)} \beta + O(\beta^2) \\ \varphi^* &\simeq_{\beta \rightarrow \infty} -\frac{1}{2\psi'(\gamma/2)\beta} + O(1/\beta^2) . \end{aligned} \quad (86)$$

For small displacement around this optimum direction $\varphi = \varphi^* + \delta\varphi$, one retrieves an isotropic continuum limit characterized by an elastic coefficient κ such that $c_\varphi \simeq c_{\varphi^*} - \frac{1}{4}\kappa\delta\varphi^2$. One easily find using (80):

$$\kappa = -8 \frac{(\psi'(\gamma/2))^2}{\psi''(\gamma/2) - \psi''(\beta + \gamma/2)} , \quad (87)$$

which generalizes the known result for the log-Gamma.

Degeneration towards the log-Gamma and Strict-Weak polymers.

- The Laplace transform of the partition sum of the log-Gamma polymer is obtained as $\overline{\exp(-uZ_t^{LG}(x))} = \lim_{\beta \rightarrow \infty} \overline{\exp(-\frac{u}{\beta} Z_t(x))}$. This amounts to change $u^s \rightarrow u^s \exp(-ts \log(\beta))$ in the above formulas. For large β we use the limits $\psi'(x) \rightarrow_{x \rightarrow \infty} 0$, $\psi''(x) \rightarrow_{x \rightarrow \infty} 0$ and $\psi(x) = \log(x) - \frac{1}{2x} + O(\frac{1}{x^2})$. It is then easily seen that the presence of β do not change the position of k_φ in this limit nor the amplitudes of the fluctuations λ_φ , whereas c_φ receives a contribution proportional to $-\log(\beta)$ which exactly cancels the rescaling of the partition sum. This shows that the system of equation (80) converges to the one of the log-Gamma.
- In the case of the Strict-Weak polymer, the rescaling of the partition sum introduces a term that amounts to change $u^s \rightarrow u^s \exp(sx \log(\gamma)) = \exp(s(1/2 + \varphi) \log(\gamma))$ in the above formulas. This suggest to look for a solution of the form $k_\varphi = -\frac{\gamma}{2} + k_\varphi^{SW}$. The system of equation (80) then converges to

$$0 = -\left(\frac{1}{2} - \varphi\right) \psi'(k_\varphi^{SW}) + \psi'(\beta + k_\varphi^{SW}) \quad (88)$$

$$c_\varphi^{SW} = \left(\frac{1}{2} - \varphi\right) \psi(k_\varphi^{SW}) - \psi(\beta + k_\varphi^{SW}) \quad (89)$$

$$\lambda_\varphi^{SW} = \left(-\frac{t}{8} \left(\left(\frac{1}{2} - \varphi\right) \psi''(k_\varphi^{SW}) - \psi''(\beta + k_\varphi^{SW}) \right) \right)^{\frac{1}{3}} , \quad (90)$$

so that we retrieve the result of⁵⁶ for the Strict-Weak polymer case (the precise correspondence with their notations reads $\kappa = 1/(1/2 - \varphi)$, $\bar{t} = k_\varphi$, $k = \beta$, $\bar{f}_{k,\kappa} = -\kappa c_\varphi^{SW}$ and $\bar{g}_{k,\kappa} = \frac{8}{t(1/2 - \varphi)} (\lambda_\varphi)^3$).

D. A low temperature limit.

1. Definition of the zero temperature model

In this section we study the limit $\gamma = \epsilon\gamma'$ and $\beta = \epsilon\beta'$ of the model with $\epsilon \rightarrow 0$ (hence, $\alpha \rightarrow 1$). As we show now, this model converges to a zero temperature problem.

The analysis is similar to⁵⁹. There (Lemma 4.1) it was shown that for a random variable z chosen with a $Beta(\alpha = \epsilon a, \beta = \epsilon b)$ distribution, the joint PDF of the pair $(-\epsilon \ln z, -\epsilon \ln(1-z))$ converges in law to $(\xi E_a, (1-\xi)E_b)$ as $\epsilon \rightarrow 0$ where ξ a Bernoulli random variable (i.e. $\xi = 0, 1$ with probabilities $p = b/(a+b), 1-p$) and E_a, E_b exponential random variables of parameters a and b respectively (i.e. $p(E) = ae^{-aE}\theta(E)$) statistically independent from ξ . Note that the correlations between E_a and E_b are unimportant since they are multiplied by ξ and $1-\xi$ which cannot be non-zero simultaneously. The occurrence of the Bernoulli variable is intuitively understood since in that limit $p(u)$ exhibits two peaks, one near $u = 0$ and one near $u = 1$ with weights p and $1-p$, and the exponential distributions arise by zooming-in on these peaks and rescaling (u for the first peak, $v = 1-u$ for the other peak).

Since in the Inverse-Beta model $1/u$ is distributed as a $Beta(\gamma, \beta)$ random variable, we immediately obtain that the rescaled random energies of the model $(\mathcal{E}_u, \mathcal{E}_v) = (-\epsilon \log(u), -\epsilon \log(v))$ converge in probability to

$$(-\epsilon \log(u), -\epsilon \log(v)) \sim_{\epsilon \rightarrow 0} (-\zeta E_{\gamma'}, (1-\zeta)E_{\beta'} - \zeta E_{\gamma'}) = (\mathcal{E}'_u, \mathcal{E}'_v), \quad (91)$$

where ζ is a Bernoulli random variable of parameter $p = \beta'/(\gamma' + \beta')$, $E_{\gamma'}$ and $E_{\beta'}$ are exponential random variables of parameter $\gamma' > 0$ and $\beta' > 0$, independent of ζ . Equivalently one can choose:

$$(\mathcal{E}'_u, \mathcal{E}'_v) = (0, E_{\beta'}) \quad , \quad \text{with proba } 1-p \quad (92)$$

$$(\mathcal{E}'_u, \mathcal{E}'_v) = (-E_{\gamma'}, 1) \quad , \quad \text{with proba } p \quad (93)$$

i.e. a model where disorder is chosen randomly either on the site or on the pair of edges arriving at it, with a penalty for the horizontal edge. The two cases corresponds to two peaks near $(u, v) = (1, 0)$ and $(u, v) = (+\infty, +\infty)$ in their distribution in that limit. In terms of the partition sum of the polymer, the limit reads

$$\begin{aligned} -\epsilon \log(Z_t(x)) &= -\epsilon \log\left(\sum_{\pi: (0,0) \rightarrow (t,x)} \exp\left(\sum_{e \in \pi} \log(w_e)\right)\right) \\ &= -\epsilon \log\left(\sum_{\pi: (0,0) \rightarrow (t,x)} \exp\left(-\frac{1}{\epsilon} \sum_{e \in \pi} E_e\right)\right) \\ &\sim_{\epsilon \rightarrow 0} \min_{\pi: (0,0) \rightarrow (t,x)} \sum_{e \in \pi} \mathcal{E}'_e \quad := \mathfrak{E}_{(t,x)} . \end{aligned} \quad (94)$$

which justifies the name zero temperature limit: the rescaled free energy of the original model converges in probability to the minimal energy $\mathfrak{E}_{(t,x)}$ for the set of all polymers with starting points $(0, 0)$ and ending points (t, x) in the random environment with energies \mathcal{E}'_e distributed according to (91).

Degeneration to the Exponential (i.e. $q = 1$) Johansson model: In the so-called log-Gamma limit, i.e. $\beta' \rightarrow +\infty$, one obtains $p = 1$ hence:

$$(\mathcal{E}'_u, \mathcal{E}'_v) = (-E_{\gamma'}, 1) \quad (95)$$

i.e. the on-site exponential distribution model of parameter γ , also identical to the $q \rightarrow 1$ limit of the Johansson model, studied in²¹. Note that the extra weight w_{00} at the origin which we included, allows to precisely recover the Johansson polymer model (with an exponential variable also on the site $x = t = 0$). To make contact with the notations of²¹ we have $\mathfrak{E}_{(t,x)} = -H(M, N)$ with $M = I = 1 + x$ and $N = J = 1 + t - x$.

Degeneration to the zero-temperature limit of the Strict-Weak model: In the limit $\gamma' \rightarrow \infty$ one obtains $p = 0$ hence:

$$(\mathcal{E}'_u, \mathcal{E}'_v) = (0, E_{\beta'}) \quad (96)$$

This model can be interpreted as a discretization of a zero temperature version of the semi-discrete polymer model where one replaces the set of independent Brownian motions by a set of independent random walks.

2. Fredholm determinant formula for the zero temperature model

In order to obtain a Fredholm determinant for the zero temperature model starting from our expressions for $g_{tx}(u) = \exp(-uZ_t(x))$, we rescale u as $u = \exp(r/\epsilon)$ with $r \in \mathbb{R}$ fixed. Indeed, one then has

$$\begin{aligned} g_{tx}(-\exp(r/\epsilon)) &= \overline{\exp\left(-\exp\frac{1}{\epsilon}(r + \epsilon \log Z_t(x))\right)} \\ &\rightarrow_{\epsilon \rightarrow 0} \overline{\theta(-r + \mathfrak{E}_{(t,x)})} \\ &= \text{Prob}(\mathfrak{E}_{(t,x)} > r) . \end{aligned} \quad (97)$$

We can thus directly write a Fredholm determinant formula for $Prob(\mathfrak{E}_{(t,x)} > r)$ by inserting $u = \exp(r/\epsilon)$ in (69). The point-wise $\epsilon \rightarrow 0$ limit of the Kernel is taken using a rescaling $s \rightarrow \epsilon s$, $a \rightarrow \epsilon a$ (so that the contour \mathcal{C} do not crosses poles when we take the limit), $k \rightarrow \epsilon k$ and $v_i \rightarrow v_i/\epsilon$ and using

$$\frac{\epsilon}{\sin(\pi \epsilon s)} \xrightarrow{\epsilon \rightarrow 0} \frac{1}{\pi s} \quad , \quad \Gamma(\epsilon x) \simeq_{\epsilon \rightarrow 0} \frac{1}{\epsilon x} + O(1) . \quad (98)$$

We thus obtain $Prob(\mathfrak{E}_{(t,x)} > r) = \text{Det}(I + K_{tx}^{T=0})$ with

$$K_{t,x}^{T=0}(v_1, v_2) = - \int_{-\infty}^{+\infty} \frac{dk}{\pi} \int_C \frac{ds}{2i\pi s} e^{sr - 2ik(v_1 - v_2) - s(v_1 + v_2)} \quad (99)$$

$$\left(\frac{\frac{s}{2} + \frac{\gamma'}{2} - ik}{-\frac{s}{2} + \frac{\gamma'}{2} - ik} \right)^{1+x} \left(\frac{\frac{s}{2} + \frac{\gamma'}{2} + ik}{-\frac{s}{2} + \frac{\gamma'}{2} + ik} \right)^{1-x+t} \left(\frac{\beta' + ik + \frac{\gamma'}{2} - \frac{s}{2}}{\beta' + ik + \frac{\gamma'}{2} + \frac{s}{2}} \right)^t .$$

where now $\tilde{C} = a + i\mathbb{R}$ with $0 < a < \gamma'$ and $K_{t,x}^{T=0} : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$. Using the same type of rescaling as above, we also obtain an analogous expression to (70) as $Prob(\mathfrak{E}_{(t,x)} > r) = \text{Det}(I + K_{tx}^{BA,T=0})$ where

$$K_{t,x}^{BA,T=0}(z, z') = \int_{2a+\tilde{a}+i\mathbb{R}} dw \frac{1}{4\pi(w-z')} \frac{1}{\pi(w-z)} e^{r(w-z)} \left(\frac{\gamma' + a - z}{\gamma' + a - w} \right)^{1+x} \left(\frac{w-a}{z-a} \right)^{1-x+t} \left(\frac{z-a+\beta'}{w-a+\beta'} \right)^t \quad (100)$$

where : $K_{t,x}^{BA,T=0} : L^2(a + \tilde{a} + i\mathbb{R}) \rightarrow L^2(a + \tilde{a} + i\mathbb{R})$ $0 < a < \gamma'$ and $0 < \tilde{a} < \gamma' - a$.

We also immediately obtain a formula analogous to our conjecture (71) as *a conjecture for the $T = 0$ model: for $I \geq J$*

$$Prob(\mathfrak{E}_{(t,x)} > r) = \frac{1}{J!} \int_{(i\mathbb{R})^J} \prod_{j=1}^J \frac{dw_j}{2i\pi} \prod_{j \neq k=1}^J (w_j - w_k) \prod_{j=1}^J \frac{e^{r(w_j-a)}}{(a-w_j)^J} \left(\frac{\gamma'}{\gamma' + a - w_j} \right)^I \left(\frac{\beta'}{w_j - a + \beta'} \right)^{I+J-2} . \quad (101)$$

with $0 < a < \gamma'$.

Limit to the Johansson model: For $\beta'/\gamma' = +\infty$ one thus finds a formula for the DP model of Johansson (i.e. with independent exponentially distributed on-site energies). It is then interesting to compare our formula with the one obtained in²¹ (formula (1.18), which reads (for $r < 0$), $I \geq J \geq 1$:

$$Prob(-\mathfrak{E}_{(t,x)} < -r) = \frac{1}{Z'_{IJ}} \int_{[0,-r]^J} \prod_{j=1}^J dx_j \prod_{1 \leq i < j \leq J} (x_i - x_j)^2 \prod_{j=1}^N x_j^{I-J} e^{-x_j} \quad (102)$$

which coincides with the CDF of the largest eigenvalue of the Laguerre Unitary Ensemble (LUE) of random matrices (the constant Z'_{IJ} simply ensures the normalisation to unity of the measure on $(\mathbb{R}^+)^J$).

3. Asymptotic analysis and KPZ universality for the zero temperature model

We now study the large length limit of the zero temperature model: $t \rightarrow \infty$ and $x = (1/2 + \varphi)t$. The analysis is similar to the one made for the finite temperature model and here we only give the main steps. As before, the $t \rightarrow \infty$ limit is dominated by a saddle point. The dominating term in the Fredholm determinant (99) now reads $\exp(t(\tilde{G}_\varphi(\frac{s}{2} + ik) - \tilde{G}_\varphi(-\frac{s}{2} + ik)))$ with

$$\tilde{G}_\varphi(y) = -(\frac{1}{2} + \varphi) \log(\frac{\gamma'}{2} - y) + (\frac{1}{2} - \varphi) \log(\frac{\gamma'}{2} + y) - \log(\beta' + \frac{\gamma'}{2} + y) . \quad (103)$$

Note that with the contour previously chosen the arguments of \tilde{G}_φ stay away from the branch cut of the logarithm. As before we look for a critical point, $(s, k) = (0, -i\tilde{k}_\varphi)$ such that $\tilde{G}_\varphi''(\tilde{k}_\varphi) = 0$. This defines \tilde{k}_φ as

$$\frac{(\frac{1}{2} + \varphi)}{(\frac{\gamma'}{2} - \tilde{k}_\varphi)^2} - \frac{(\frac{1}{2} - \varphi)}{(\frac{\gamma'}{2} + \tilde{k}_\varphi)^2} + \frac{1}{(\beta' + \frac{\gamma'}{2} + \tilde{k}_\varphi)^2} = 0. \quad (104)$$

Note that this equation as in general several solutions, but the only physical one must have $|\tilde{k}_\varphi| < \gamma'/2$ to truly dominate the integration. To this point, we can now follow the exact same steps as before by taking

$$\begin{aligned} r &= t\tilde{c}_\varphi - \tilde{\lambda}_\varphi \tilde{z} \\ \tilde{c}_\varphi &= -\tilde{G}'_\varphi(\tilde{k}_\varphi) \quad , \quad \tilde{\lambda}_\varphi = \left(\frac{t\tilde{G}'''_\varphi(\tilde{k}_\varphi)}{8} \right)^{\frac{1}{3}} \end{aligned} \quad (105)$$

and using the same rescalings in (99). In the large length limit, this leads to

$$\lim_{t \rightarrow \infty} \text{Prob} \left(\frac{\mathfrak{E}_{(t,x=(1/2+\varphi)t)} - t\tilde{c}_\varphi}{\tilde{\lambda}_\varphi} > -2^{\frac{2}{3}} \tilde{z} \right) = F_2(\tilde{z}) \quad (106)$$

with

$$\tilde{c}_\varphi = -\frac{(\frac{1}{2} + \varphi)}{\frac{\gamma'}{2} - \tilde{k}_\varphi} - \frac{(\frac{1}{2} - \varphi)}{\frac{\gamma'}{2} + \tilde{k}_\varphi} + \frac{1}{\beta' + \frac{\gamma'}{2} + \tilde{k}_\varphi} \quad (107)$$

$$0 = \frac{(\frac{1}{2} + \varphi)}{(\frac{\gamma'}{2} - \tilde{k}_\varphi)^2} - \frac{(\frac{1}{2} - \varphi)}{(\frac{\gamma'}{2} + \tilde{k}_\varphi)^2} + \frac{1}{(\beta' + \frac{\gamma'}{2} + \tilde{k}_\varphi)^2} \quad (108)$$

$$\tilde{\lambda}_\varphi = \left(\frac{t}{8} \left(\frac{(1+2\varphi)}{(\frac{\gamma'}{2} - \tilde{k}_\varphi)^3} + \frac{(1-2\varphi)}{(\frac{\gamma'}{2} + \tilde{k}_\varphi)^3} - \frac{2}{(\beta' + \frac{\gamma'}{2} + \tilde{k}_\varphi)^3} \right) \right)^{\frac{1}{3}}. \quad (109)$$

Note that this result is coherent with the one obtained at finite temperature (79) and (80) and can be obtained from it by scaling $\gamma = \epsilon\gamma'$, $\beta = \epsilon\beta'$ and $k_\varphi = \epsilon\tilde{k}_\varphi$.

Angle of optimal energy The angle of minimum energy $\tilde{\varphi}^*$ of the model is obtained by solving $\frac{\partial}{\partial \varphi} \tilde{c}_\varphi = 0$. This imposes $\tilde{k}_\varphi = 0$ and, using (107), we thus obtain

$$\varphi^* = -\frac{\gamma'^2}{8} \frac{1}{(\beta' + \frac{\gamma'}{2})^2} < 0. \quad (110)$$

As for the finite temperature model, we thus retrieve that $\beta' > 0$ biases the DP towards the vertical direction. For $\beta' \rightarrow 0$ we obtain once again $\varphi^* = -\frac{1}{2}$. The optimal energy per unit length, and the scaling parameter $\tilde{\lambda}_{\varphi^*}$ at the optimal angle are respectively

$$\tilde{c}_{\varphi^*} = -\frac{2\beta'}{\gamma'(\beta' + \frac{\gamma'}{2})} \quad , \quad \tilde{\lambda}_{\varphi^*} = \left(\frac{2}{(\gamma')^3} - \frac{2}{(\gamma' + 2\beta')^3} \right)^{\frac{1}{3}} t^{\frac{1}{3}} \quad (111)$$

recovering the results for the Johansson model:

In the limit $\beta' = +\infty$ the above equations (107) can be solved explicitly. One finds $\tilde{k}_\varphi = -\frac{\gamma'}{4\varphi}(1 - \sqrt{1 - 4\phi^2})$, where we have chosen the root which vanished at the optimal angle $\varphi^* = 0$ (i.e. the diagonal which is a symmetry axis in this case). This yields:

$$\tilde{c}_\varphi = -\frac{1}{\gamma'}(1 + \sqrt{1 - 4\phi^2}) \quad , \quad \tilde{\lambda}_\varphi = \frac{1}{\gamma'} t^{\frac{1}{3}} \left(\frac{8\varphi^4}{(1 - \sqrt{1 - 4\phi^2})^2 \sqrt{1 - 4\phi^2}} \right)^{\frac{1}{3}} \quad (112)$$

We can now compare with Johansson result (formula 1.22 in²¹) which reads (for $\gamma' = 1$):

$$H(gJ, J) \simeq_{J \rightarrow +\infty} (1 + \sqrt{g})^2 J + g^{-1/6} (1 + \sqrt{g})^{4/3} J^{1/3} \chi_2 \quad (113)$$

where χ_2 is a Tracy-Widom GUE random variable (of CDF given by F_2). With a little bit of algebra one can check that this is exactly equivalent to our result, namely:

$$\mathfrak{E}_{(t,x=(1/2+\varphi)t)} \simeq_{t \rightarrow +\infty} t\tilde{c}_\varphi - 2^{2/3} \lambda_\varphi \chi_2 \quad (114)$$

with $\mathfrak{E}_{(t,x=(1/2+\varphi)t)} = -H(gJ, J)$, taking into account that $J = 1 + t - x \simeq (\frac{1}{2} - \phi)t$, hence $g = \frac{1+2\varphi}{1-2\varphi}$.

V. CONCLUSION

In this paper we attempted a classification of finite temperature directed polymer models on the square lattice with homogeneously distributed random Boltzmann weights and a certain type of short-range correlations (Section II A), for which the moments of the partition sum $Z_t(x)$ can be calculated via a coordinate Bethe ansatz. Following the pioneering work of²⁴, we obtained a rigorous expression (29) that constrains the possible forms for the moments of the underlying distribution of weights. We discussed in details the possibilities of finding PDF's with the appropriate moments (29) and, though the classification is still not complete, we were able to exclude a large number of cases. In cases where the moment problem has a solution, we retrieved all the previously known finite temperature integrable DP models (Section III B), and introduced a new one, the Inverse-Beta polymer, which appears as a natural two parameters generalization of the log-Gamma polymer, but also contains the Strict-Weak polymer as a limit. Using the Bethe ansatz, we obtained an integral formula for the moments of the partition sum (60) of the Inverse-Beta polymer, with point-to-point boundary conditions. Along this route, most of the tools developed in⁵⁵ for the Bethe ansatz solution of the log-Gamma polymer proved very useful and were generalized.

Starting from the moments formula and using analytical continuations, we obtained two equivalent Fredholm determinant formulas for the Laplace transform of the PDF of the partition sum (69) and (70), and conjectured a n -fold integral formula (71) for the same object, which generalizes a known formula for the log-Gamma polymer obtained in⁵³ in the framework of the gRSK correspondence. Using our Fredholm determinant formulas and an asymptotic analysis in the limit of large polymer length, we were able to obtain the KPZ universality of the model (critical exponents and Tracy-Widom GUE free-energy fluctuations) (79) and as well as exact implicit expressions for the mean free energy and the amplitude of fluctuations as a function of the polymer orientation w.r.t. the diagonal. As an application we obtained an exact expression for the optimal angle which minimizes the free-energy of the polymer (83).

In Section IV D we introduced a zero-temperature DP model as a limit of the Inverse-Beta polymer, which generalizes the previously known zero-temperature limit of the log-Gamma polymer. Using the exact formulas obtained for the Inverse-Beta polymer, we showed analogous formulas for this zero-temperature model. In particular we obtained exact formulas (Fredholm determinant and n -fold integrals) for the cumulative distribution of optimal energy of the model (99), (100) and (101). Using an asymptotic analysis, we showed the KPZ universality (106) of the model. Our formula compare successfully with some results obtained by Johansson in his pioneering study of the Exponential zero-temperature polymer²¹, a particular case of our zero-temperature model.

We believe that the present work could be used as a guide for future research of new integrable DP models. In addition, we once again showed that the replica Bethe ansatz method is a valuable and versatile tool for the analysis of such DP models. In particular, some results of this paper could prove useful and adaptable to the analysis of the model with different boundary conditions.

For future works on the Inverse-Beta polymer, it should be very interesting to obtain a solution of this model using the gRSK correspondence or a generalization of the latter (as in the recent work⁶²). Our conjecture (71) could be proven (or invalidated) using these techniques. In addition, we know that an inhomogeneous version of the log-Gamma polymer was amenable to analytical treatment in the framework of the gRSK correspondence, and it is thus likely that an inhomogeneous version of the Inverse-Beta model should also exist.

For future works on the classification of directed polymer models, various directions of research remain. The most direct one is to understand if some integrable models remain to be found to fill the left voids in Fig. 2 (as e.g. our proposal of Appendix B). Other directions would be to extend this framework to introduce inhomogeneous models, or different disorder correlations. The precise implications of our classification of finite temperature DP model for possible zero temperature integrable DP models remain to be elucidated. Indeed all the models we found in our framework admit a zero temperature limit. For example, the zero temperature limit of the log-Gamma model is the $q \rightarrow 1$ limit (beware that this q is a priori different from the one used in Section III) of the zero temperature model of Johansson²¹, i.e. the Exponential zero-T model (as was pointed out in⁵³). However, at this stage, our framework seems to miss the $q \neq 1$ case of the Johansson model. A natural question is then to understand if a finite temperature integrable DP model sits above the Johansson model $\forall q$, and whether the zero temperature model studied in this paper admits a $q \neq 1$ generalization. Since Johansson's model is determinantal, a related outstanding question is to obtain a deeper and more systematic understanding of the relations between Bethe ansatz solvable models and determinantal processes which seem to often occur as limit cases of the former.

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Appendix A: The $|q| < 1$ case: study of degenerations.

Here we study in details the possible degenerations of the parameters (q, ν, μ) that would eventually lead to a PDF $p(u, v)$ such that the moments of u and v are given by (29) and $n_{\max} \geq 2$. As in the main text, we restrict to the domain $|q| < 1$ and consider the random variable $z_x = u + xv$, $x \in \mathbb{R}$. Its variance is:

$$\overline{z_x^2}^c = \frac{(\mu - 1)(1 - q)(\mu - \nu)(\mu x - 1)(\mu x - \nu)}{\mu^2(\nu - 1)^2(\nu q - 1)} \quad (\text{A1})$$

which must be positive. Since the polynomial in x changes sign at $x = \frac{\nu}{\mu}$ and $x = \frac{1}{\mu}$ one must look for cases where $\frac{\nu}{\mu} = \frac{1}{\mu}$. The different cases to investigate are thus $\nu = 1$, $|\mu| = \infty$, $\mu = 0$ and a combination of these cases. It is instructive to look at the variance for $x = 1$:

$$\overline{z_1^2}^c = \frac{(\mu - 1)^2(1 - q)(\mu - \nu)^2}{\mu^2(\nu - 1)^2(\nu q - 1)} \quad (\text{A2})$$

One sees that the positivity of the variance implies $q\nu > 1$, as long as we do not consider degenerations $\mu \rightarrow 1$, $\mu \rightarrow \nu$, or $q \rightarrow 1$ (in which case the variance of z_1 may vanish and the condition may disappear).

- If $\nu \rightarrow 1$, it is easy to see from the variance of z_1 that there must be at least one additional degeneration, either (i) $\mu \rightarrow 1$ or (ii) $q \rightarrow 1$ (iii) both $q \rightarrow 1$ and $\mu \rightarrow 1$. The first one can be ruled out as follows: setting $\mu = 1 + a\epsilon$ and $\nu = 1 + b\epsilon$ with $\epsilon \rightarrow 0$, one finds that the variance is $\overline{z_x^2}^c = \frac{a(b-a)}{b^2}(1-x)^2$ and that from (29) the marginals have integer moments $\overline{u^n} = 1 - \frac{a}{b}$ and $\overline{v^n} = \frac{a}{b}$ for all $n \geq 1$. This implies that $u + v = 1$ and $v = 0, 1$ with probability $\frac{a}{b}$, which then predicts joints moments different from the ones obtained from (29) in that limit, hence no joint PDF exists in case (i). The case (ii) and (iii) both imply a $q \rightarrow 1$ limit which we discuss in the end of this appendix.
- If $|\mu| \rightarrow \infty$. In this case, looking at the original moments (29), we see that we must scale $v \rightarrow v/\mu$ to obtain a well defined random variable. We define $(u', v') = (u, v/\mu)$. In the limit $\mu \rightarrow \infty$, the moments are

$$\overline{u'^{n_1} v'^{n_2}} = \frac{(-1)^{n_2} q^{\frac{n_2(n_2-1)}{2}}}{(\nu; q)_{n_1+n_2}} \frac{(q; q)_{n_1+n_2}}{(q; q)_{n_1} (q; q)_{n_2}} \frac{1}{C_{n_1+n_2}^{n_1}} \quad (\text{A3})$$

we now define the random variable $z'_x = u' + xv'$ and compute its variance:

$$\overline{(z'_x)^2}^c = \frac{(1 - q)(x - 1)(x - \nu)}{(\nu - 1)^2(\nu q - 1)} \quad (\text{A4})$$

which must also be positive for all x . However, this polynomial changes sign at $x = 1$ and $x = \nu$ so we must have $\nu = 1$. Since the constraint $q\nu > 1$ still holds and $q < 1$, the only possibility is to have $q \rightarrow 1$ as well, a case discussed below.

- If $\mu = 0$, looking at (29), we see that we must now rescale (u, v) as $(u', v') = (\mu u, v)$ and we obtain

$$\overline{u'^{n_1} v'^{n_2}} = \frac{(-1)^{n_1} q^{\frac{n_1(n_1-1)}{2}}}{(\nu; q)_{n_1+n_2}} \frac{(q; q)_{n_1+n_2}}{(q; q)_{n_1} (q; q)_{n_2}} \frac{1}{C_{n_1+n_2}^{n_1}} \quad (\text{A5})$$

i.e. this is a case identical to the previous one, and we also conclude that we must have $q \rightarrow 1$.

Let us now discuss all the possibilities in the $q \rightarrow 1$ limit. Taking the limit directly on (29), one obtains

$$\overline{u^{n_1} v^{n_2}} = \frac{(1 - \nu/\mu)^{n_1} (1 - \mu)^{n_2}}{(1 - \nu)^{n_1+n_2}} \quad (\text{A6})$$

where we used that at fixed n, a , $(q^a; q)_n \simeq_{q \rightarrow 1} (1 - q)^n (a)_n$, where $(a)_n = a(a+1) \dots (a+n-1)$, and here we took $a = 1$. Obviously, the limit we took only works if ν , μ and ν/μ are all different from 1, but it also encompasses other limits such that the $\mu \rightarrow \infty$ and $q \rightarrow 1$ case discussed when analyzing (A3). The moments (A6) correspond to deterministic weights $u = \frac{1-\nu/\mu}{1-\nu}$ and $v = \frac{1-\mu}{1-\nu}$. These models are obviously integrable, but trivial.

We must thus study the $q \rightarrow 1$ limit with at least one of those parameters that goes to 1. The question of the speed of the convergence then arises. In general, taking $q = 1 - \epsilon$ and $a = 1 - a'\epsilon^\zeta$ with $\zeta > 0$, one has $(a; q)_n \simeq_{q \rightarrow 0} \epsilon^n (a')_n$

(if $\zeta = 1$), $(a; q)_n \simeq_{q \rightarrow 0} \epsilon^\zeta a' \epsilon^{n-1} (n-1)!$ (if $\zeta > 1$ and $n \geq 1$) and $(a; q)_n \simeq_{q \rightarrow 0} \epsilon^{n\zeta} (a')^n$ (if $\zeta < 1$). The possibility of using $\zeta < 1$ for the convergence of μ and/or ν (slow convergence compared to q) is uninteresting since it leads to pure power-laws. The possibility of using $\zeta > 1$ (fast convergence compared to q) is also uninteresting since one cannot rescale (u, v) to obtain well-defined moments in the $\epsilon \rightarrow 0$ limit. In the following, we thus only consider the possibility of convergence of the parameters *at the same speed* than q . Let us first examine the cases where only one of those parameters goes to 1. We obtain

- If $\mu = q^\beta$ and $\nu \neq 1$,

$$\overline{u^{n_1} v^{n_2}} = (\beta)_{n_2} \times \text{power} - \text{laws} \quad (\text{A7})$$

- If $\nu = q^{\alpha+\beta}$ and $\mu \neq 1$,

$$\overline{u^{n_1} v^{n_2}} = \frac{1}{(\alpha + \beta)_{n_1+n_2}} \times \text{power} - \text{laws} \quad (\text{A8})$$

- If $\nu/\mu = q^\alpha$, $\nu \neq 1$ and $\mu \neq 1$,

$$\overline{u^{n_1} v^{n_2}} = (\alpha)_{n_1} \times \text{power} - \text{laws} \quad (\text{A9})$$

where we have not written the precise form of the unimportant power-law terms. As discussed in the main text (Section III B), these cases indeed correspond to proper PDF's and known integrable models for some range of parameters α , β . The first and third ones correspond to the moments of the Strict-Weak polymers, the second one corresponds to the moments of the log-Gamma polymer. Notice however that these models can all be obtained as limits of the case $(q, \mu, \nu) \rightarrow (1, 1, 1)$. Indeed, taking $\mu = q^\beta$, $\nu = q^{\alpha+\beta}$ and $q \rightarrow 1$, we obtain

$$\overline{u^{n_1} v^{n_2}} = \frac{(\alpha)_{n_1} (\beta)_{n_2}}{(\alpha + \beta)_{n_1+n_2}}. \quad (\text{A10})$$

Taking appropriate limits on this last formula, we can retrieve the three precedent cases. For example the limit $\beta \rightarrow \infty$ of (A10) leads to, rescaling u as βu , $(\beta u)^{n_1} v^{n_2} = (\alpha)_{n_1}$, and we obtain the moments of the third case. Taking a similar limit $|\alpha|, |\beta| \rightarrow \infty$ with $\alpha + \beta$ fixed, we obtain the second case.

Hence, in this sense, the most general limit of (29) that can lead to distributions with a well-defined variance is the $(q, \mu, \nu) \rightarrow (1, 1, 1)$ limit. This limit is studied in details in Section III B.

Appendix B: A more systematic study of analytical continuations of moments.

Starting with a polymer model with moments given by

$$\overline{u^{n_1} v^{n_2}} = (\epsilon_1)^{n_1} (\epsilon_2)^{n_2} \frac{(\alpha)_{n_1} (\beta)_{n_2}}{(\alpha + \beta)_{n_1+n_2}}, \quad (\text{B1})$$

where (α, β) are arbitrary, $(\epsilon_1, \epsilon_2) \in \{-1, 1\}^2$, we look for distributions $p(u, v)$ such that (B1) corresponds to the moments of positive random variables. It is natural, in agreement with the examples of the Beta and Inverse-Beta polymers studied in the text, to analytically continue (B1) to $(n_1, n_2) = (s_1, s_2) \in \mathbb{C}^2$. Obviously there is an infinite number of possible analytical continuations and here we only study arguably the most natural ones (using Euler inversion formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ and $(-1)^n = \sin(\pi(x+n))/\sin(\pi x)$ for integer n), which we now enumerate.

- First type, $(\epsilon_1, \epsilon_2) = (1, 1)$:

$$\overline{u^{s_1} v^{s_2}} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha + s_1)\Gamma(\beta + s_2)}{\Gamma(\alpha + \beta + s_1 + s_2)}. \quad (\text{B2})$$

- Second type $(\epsilon_1, \epsilon_2) = (1, -1)$:

$$\overline{u^{s_1} v^{s_2}} = \frac{\Gamma(1 - \alpha)}{\Gamma(1 - \alpha - \beta)\Gamma(\beta)} \frac{\Gamma(1 - \alpha - \beta - s_1 - s_2)\Gamma(\beta + s_2)}{\Gamma(1 - \alpha - s_1)}. \quad (\text{B3})$$

- Third type $(\epsilon_1, \epsilon_2) = (-1, -1)$:

$$\overline{u^{s_1} v^{s_2}} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha-\beta)} \Gamma(\alpha+s_1)\Gamma(\beta+s_2)\Gamma(1-\alpha-\beta-s_1-s_2) . \quad (\text{B4})$$

- Fourth type, $(\epsilon_1, \epsilon_2) = (-1, 1)$:

$$\overline{u^{s_1} v^{s_2}} = \frac{\Gamma(\alpha+\beta)\Gamma(1-\alpha)}{\Gamma(\beta)} \frac{\Gamma(\beta+s_2)}{\Gamma(1-\alpha-s_1)\Gamma(\alpha+\beta+s_1+s_2)} . \quad (\text{B5})$$

- Fifth type $(\epsilon_1, \epsilon_2) = (-1, -1)$:

$$\overline{u^{s_1} v^{s_2}} = \Gamma(\alpha+\beta)\Gamma(1-\alpha)\Gamma(1-\beta) \frac{1}{\Gamma(1-\alpha-s_1)\Gamma(1-\beta-s_2)\Gamma(\alpha+\beta+s_1+s_2)} . \quad (\text{B6})$$

- Sixth type $(\epsilon_1, \epsilon_2) = (1, 1)$:

$$\overline{u^{s_1} v^{s_2}} = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(1-\alpha-\beta)} \frac{\Gamma(1-\alpha-\beta-s_1-s_2)}{\Gamma(1-\alpha-s_1)\Gamma(1-\beta-s_2)} . \quad (\text{B7})$$

a. First type

Let us first consider the first type of analytical continuation (B2). Assuming it to be valid on the full complex plane, the distribution $p(u, v)$ can be directly obtained as an ILT

$$p(u, v) = \frac{1}{N_{\alpha, \beta}} \int_{\mathcal{C}_1} \frac{ds_1}{2i\pi} \int_{\mathcal{C}_2} \frac{ds_2}{2i\pi} u^{-1-s_1} v^{-1-s_2} \frac{\Gamma(\alpha+s_1)\Gamma(\beta+s_2)}{\Gamma(\alpha+\beta+s_1+s_2)} \quad (\text{B8})$$

where $N_{\alpha, \beta}^{-1} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ is a normalization factor, and different contours \mathcal{C}_i can be considered. Here we first consider the most natural choices: vertical lines passing through the right of all the poles of the integrands located at $s_1 = -\alpha - m_1$ and $s_2 = -\beta - m_2$ with $(m_1, m_2) \in \mathbb{N}^2$, e.g. $s_1 = -\alpha + 1 + iy_1$ and $s_2 = -\beta + 1 + iy_2$ with $(y_1, y_2) \in \mathbb{R}^2$. We first consider the integration on s_2 . If $v > 1$, the contour can be closed to the right, giving 0 as a result. On the other hand, if $v < 1$, the contour can only be closed to the left and all the poles of s_2 contribute. This shows

$$p(u, v) = \theta(0 < v < 1) \frac{1}{N_{\alpha, \beta}} \int_{\mathcal{C}_1} \frac{ds_1}{2i\pi} u^{-1-s_1} \sum_{m_2=0}^{\infty} v^{-1+\beta+m_2} \frac{(-1)^{m_2}}{m_2!} \frac{\Gamma(\alpha+s_1)}{\Gamma(\alpha+s_1-m_2)} . \quad (\text{B9})$$

In this expression, one recognizes the Taylor expansion

$$(1-v)^\eta = \sum_{k=0}^{\infty} (-v)^k \frac{\Gamma(\eta+1)}{\Gamma(1+k)\Gamma(\eta-k+1)} = \sum_{k=0}^{\infty} \frac{(-v)^k}{k!} \frac{\Gamma(\eta+1)}{\Gamma(\eta-k+1)} , \quad (\text{B10})$$

with $\eta = \alpha + s_1 - 1$, and the convergence is here assured by the fact that $v < 1$. We thus get

$$p(u, v) = \theta(0 < v < 1) \frac{1}{N_{\alpha, \beta}} \int_{\mathcal{C}_1} \frac{ds_1}{2i\pi} u^{-1-s_1} v^{-1+\beta} (1-v)^{\alpha+s_1-1} . \quad (\text{B11})$$

And one now recognizes the integral representation of the Dirac δ distribution: $\int_{\mathcal{C}_1} \frac{ds_1}{2i\pi} w^{s_1} = \delta(w-1)$, and

$$\begin{aligned} p(u, v) &= \theta(0 < v < 1) \frac{1}{N_{\alpha, \beta}} u^{-1} v^{-1+\beta} (1-v)^{\alpha-1} \delta\left(\frac{1-v}{u} - 1\right) \\ &= \theta(0 < u < 1) \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{-1+\alpha} v^{-1+\beta} \delta(u+v-1) . \end{aligned} \quad (\text{B12})$$

Which is exactly the Beta distribution of the Beta polymer, and the normalizability condition imposes $\alpha > 0$ and $\beta > 0$.

b. Second type

Introducing as in the main text $\gamma = 1 - \alpha - \beta$, we now study B3, which reads in these variables:

$$\overline{u^{s_1} v^{s_2}} = \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \frac{\Gamma(\gamma - s_1 - s_2)\Gamma(\beta + s_2)}{\Gamma(\gamma + \beta - s_1)}. \quad (\text{B13})$$

We follow the same step as before

$$p(u, v) = \frac{1}{\tilde{N}_{\gamma, \beta}} \int_{\mathcal{C}_1} \frac{ds_1}{2i\pi} \int_{\mathcal{C}_2} \frac{ds_2}{2i\pi} u^{-1-s_1} v^{-1-s_2} \frac{\Gamma(\gamma - s_1 - s_2)\Gamma(\beta + s_2)}{\Gamma(\gamma + \beta - s_1)} \quad (\text{B14})$$

where $\tilde{N}_{\gamma, \beta}^{-1} = \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)}$. The contours \mathcal{C}_1 and \mathcal{C}_2 are chosen so as to avoid the poles $s_1 = \gamma - s_2 + m_1$ and $s_2 = -\beta - m_2$ for $(m_1, m_2) \in \mathbb{N}^2$, e.g. we choose $s_2 = -\beta + 1 + iy_2$ and $s_1 = \gamma + \beta - 2 + iy_2$ with $(y_1, y_2) \in \mathbb{R}^2$. Integrating first on s_1 and following the same steps as before, we now obtain

$$\begin{aligned} p(u, v) &= \theta(1 < u) \frac{1}{\tilde{N}_{\gamma, \beta}} \int_{\mathcal{C}_2} \frac{ds_2}{2i\pi} v^{-1-s_2} \sum_{m_1=0}^{\infty} \frac{(-1)^{m_1}}{m_1!} u^{-1-\gamma+s_2-m_1} \frac{\Gamma(\beta + s_2)}{\Gamma(\beta + s_2 - m_1)} \\ &= \frac{\theta(1 < u)}{\tilde{N}_{\gamma, \beta}} \int_{\mathcal{C}_2} \frac{ds_2}{2i\pi} v^{-1-s_2} u^{-1-\gamma+s_2} \left(1 - \frac{1}{u}\right)^{\beta+s_2-1} \\ &= \frac{\theta(1 < u)}{\tilde{N}_{\gamma, \beta}} v^{-1} u^{-1-\gamma} \left(1 - \frac{1}{u}\right)^{\beta-1} \delta\left(\frac{u}{v} \left(1 - \frac{1}{u}\right) - 1\right) \\ &= \frac{\theta(1 < u)}{\tilde{N}_{\gamma, \beta}} u^{-1-\gamma} \left(1 - \frac{1}{u}\right)^{\beta-1} \delta(v - 1 + u) \end{aligned}$$

We thus obtain the same distribution as before, which is indeed normalizable for $\gamma > 0$ and $\beta > 0$. Note the interesting fact that, though the moments of the distribution only exist for $n_1 + n_2 < \gamma$, the analytical continuation of the moments offered by the Gamma function allows us to retrieve the distribution. Though non-rigorous it gives some insight to understand why the replica method developed in this paper to retrieve the PDF of the partition sum of the associated polymer model works. This is also in agreement with Appendix D

c. Third type

For the third type (B4), it seems difficult to compute the involved integrals in full generality since they depend on the precise position of the poles. However, we now directly exhibit some examples that define proper distributions for $1 - (\alpha + \beta) = \gamma > 0$. We take $u = \tilde{u}w, v = \tilde{v}w$ with \tilde{u}, \tilde{v} and w independent random variables distributed with PDF

$$\begin{aligned} p_w(w) &= \frac{1}{\Gamma(\gamma)} w^{-1-\gamma} e^{-1/w} \\ p_u(u) &= \frac{1}{\Gamma(\alpha)} u^{-1+\alpha} \left(e^{-u} - \left(\sum_{k=0}^{\lfloor -\alpha \rfloor} \frac{u^k}{k!} \right) \right) \\ p_v(v) &= \frac{1}{\Gamma(\beta)} u^{-1+\beta} \left(e^{-u} - \left(\sum_{k=0}^{\lfloor -\beta \rfloor} \frac{u^k}{k!} \right) \right) \end{aligned} \quad (\text{B15})$$

Where $\lfloor (\cdot) \rfloor$ denotes the integer part, and the sum appearing in p_u (resp. p_v) is present only if $\alpha < 0$ (resp. $\beta < 0$) and regularizes the eventual divergences near the origin. These distributions are singular and only have a few integer moments, but their complex moments $\overline{u^{s_1} v^{s_2}}$ do exist on a domain $\text{Re}(s_1 + s_2) \leq \gamma$, supplemented by the condition $|\text{Re}(s_1)| < 1/2$ (resp. $|\text{Re}(s_2)| < 1/2$) if $\alpha < 0$ (resp. $\beta < 0$), and are there given by (B4). As in the log-Gamma and Inverse-Beta cases, these moments can be analytically continued to the full complex plane, opening a way for a Bethe ansatz solution of this kind of model. In terms of contours integrals, p_u can be obtained using the same technique as before with $u^s = \Gamma(\alpha + s_1)/\Gamma(\alpha)$, but always choosing a contour of integration as a vertical line passing by the origin (and eventually separating the poles of the integrand). It would be of great interests to understand if one can obtain exact results for a polymer model defined with these types of weights (e.g. the PDF of $\log Z_t(x)$) using analytical continuations of other known results. This is left for future work. Notice that these models could well be good candidates to fill the void left in the down-left quarter of Fig. 2.

d. *Other types*

For the other types, one intuitively see that they “lack of poles” in the complex plane $(s_1, s_2) \in \mathbb{C}^2$ to obtain a meaningful result after Laplace inversion, and the corresponding integrals diverge. For the fourth (B5) and fifth cases (B6), another argument goes in the same direction. Writing schematically $\overline{u^{s_1} v^{s_2}} = f(s_1, s_2)$, we have

$$\overline{(\ln u)^2}^c = \frac{\partial^2}{\partial s_1^2} f|_{s_1=s_2=0} \quad , \quad \overline{(\ln v)^2}^c = \frac{\partial^2}{\partial s_2^2} f|_{s_1=s_2=0} \quad , \quad (\text{B16})$$

where $\overline{(\cdot)}^c$ denotes the variance. Applying this formula on (B5) and (B6) always leads to negative results which is incompatible with $\ln u$ having a PDF with a second moment. We do not consider here the possibility of such very singular distributions.

For the sixth type (B7), there remains small windows of parameters for which both $\overline{(\ln u)^2}^c$ and $\overline{(\ln v)^2}^c$ are positive simultaneously, so that this argument is inconclusive. We do not investigate further the possibility of the existence of another integrable model here.

Appendix C: Effect of an additional inverse Gamma weight at the starting point

Consider the two partition sums, one, noted $Z_t(x)$ and studied in the text, in presence of the additional inverse Gamma random variable w_{00} on the site $x = t = 0$, and the other one, $\tilde{Z}_t(x)$, in absence of such a weight (which in a sense is the true point to point problem). Clearly one has:

$$Z_t(x) = w_{00} \tilde{Z}_t(x) \quad (\text{C1})$$

for any x, t where w_{00} and $\tilde{Z}_t(x)$ are uncorrelated random variables.

There are various ways to express one problem into the other. Let us use here the shorthand notation $Z \equiv Z_t(x)$ and $\tilde{Z} \equiv \tilde{Z}_t(x)$. The moments are related as:

$$\overline{Z^s} = \frac{\Gamma(\gamma - s)}{\Gamma(\gamma)} \overline{\tilde{Z}^s} \quad (\text{C2})$$

And the Laplace transforms as:

$$\overline{e^{-uZ}} = \overline{e^{-uw_{00}\tilde{Z}}} = \frac{2}{\Gamma(\gamma)} \overline{(u\tilde{Z})^{\gamma/2} K_\gamma(2\sqrt{u\tilde{Z}})} \quad (\text{C3})$$

Since the l.h.s. is known explicitly as a Fredholm determinant, we see that to obtain $P(\tilde{Z})$ one needs to invert a modified type of Laplace transform involving Bessel functions.

There is also a useful relation between the CDF's. Let us define the CDF of $\ln Z$, as $F(y) = \text{Prob}(\ln Z < y)$, and the one of $\ln \tilde{Z}$, as $\tilde{F}(y) = \text{Prob}(\ln \tilde{Z} < y)$. Clearly

$$F(y) = \text{Prob}(\ln Z < y) = \langle \text{Prob}(\ln \tilde{Z} < y - \ln w_{00}) \rangle_{w_{00}} = \langle \tilde{F}(y - \ln w_{00}) \rangle_{w_{00}} = \langle e^{-\ln w_{00} \partial_y} \rangle_{w_{00}} \tilde{F}(y) \quad (\text{C4})$$

$$= \frac{\Gamma(\gamma + \partial_y)}{\Gamma(\gamma)} \tilde{F}(y) \quad (\text{C5})$$

Hence, knowing $F(y)$ from Fredholm determinants, one can obtain the CFD $\tilde{F}(y)$ as:

$$\tilde{F}(y) = \frac{\Gamma(\gamma)}{\Gamma(\gamma + \partial_y)} F(y) \quad (\text{C6})$$

an operator which can be interpreted in the sense of a Taylor expansion w.r.t. ∂_y . At fixed γ in the large time limit studied in IV C, the rescaling (76) renders the term ∂_y smaller by $t^{-1/3}$. Defining as in (79)

$$F_{res}(z) = \text{Prob}(2^{-\frac{2}{3}} \frac{\log Z + tc_\varphi}{\lambda_\varphi} < z) = F(y = 2^{2/3} \lambda_\varphi z - tc_\varphi) \quad (\text{C7})$$

and similarly for $\tilde{F}_{res}(z)$ w.r.t. $\log \tilde{Z}$, we obtain:

$$\tilde{F}_{res}(z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma + 2^{-2/3} \lambda_\varphi^{-1} \partial_z)} F_{res}(z) \quad (\text{C8})$$

a relation exact for all t , but which for $t \rightarrow +\infty$ shows that the effect of the operator ∂_y becomes negligible in the scaling variable z (we recall that $\lambda_\varphi \sim t^{1/3}$). The rescaled CDF's are thus the same, a very intuitive result: the large-length is insensitive to such a change in the energy of the first site. The formula allows to calculate the subleading corrections.

In the $T = 0$ limit the formula (C6) simplifies. Defining $F_{T=0}(r) = \text{Prob}(\mathfrak{E}_{(t,x)} > r)$ and similarly for $\tilde{F}_{T=0}(r)$ in presence of the additional exponentially distributed energy random variable at site $(x, t) = (0, 0)$, we obtain from the definition (94):

$$\tilde{F}_{T=0}(r) = (1 - \frac{1}{\gamma'} \partial_r) F_{T=0}(r) \quad (\text{C9})$$

valid for arbitrary t . At large time the same argument on the rescaled variable \tilde{z} again shows the derivative term to be negligible.

Finally note that such relations have been studied also in the context of stationary models in the KPZ class^{9,38} where it seems also mandatory to add an inverse Gamma variable at the origin in order to obtain a FD representation. Its occurrence in a point to point problem is, to our knowledge, new.

Appendix D: Laplace transform Vs Moment Generating function: recall

In this appendix we briefly recall the idea, discussed in⁵⁵ and to which we refer for more details, that leads to the conjecture (69). It is best illustrated on the simple problem of obtaining the Laplace transform of (1):

$$\begin{aligned} g(\lambda) &= \overline{e^{-\lambda u}} = \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \int_1^{+\infty} du e^{-\lambda u} \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} \\ \overline{e^{-\lambda u}} &= \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \int_1^{+\infty} du \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} u^n \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} \end{aligned} \quad (\text{D1})$$

In this formula, it is obvious that one cannot invert the sum and integrals sign because the different terms converge only if $n < \gamma$. In which case

$$\frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \int_1^{+\infty} du \frac{(-\lambda)^n}{n!} u^n \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} = \frac{(-\lambda)^n}{n!} \frac{\Gamma(\gamma + \beta)\Gamma(\gamma - n)}{\Gamma(\gamma + \beta - n)\Gamma(\gamma)}. \quad (\text{D2})$$

Note however that, using the analytical continuation of the Gamma function, the right hand side of (D1) also makes sense for $n > \gamma$. We can thus consider the object, called the “moment generating function” defined as

$$g_{mom}(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(1+n)} \frac{\Gamma(\gamma + \beta)\Gamma(\gamma - n)}{\Gamma(\gamma + \beta - n)\Gamma(\gamma)}. \quad (\text{D3})$$

The question is now to understand how (D3) and (D1) are related. Let us now rewrite (D1) using an integral representation of the exponential as

$$\begin{aligned} g(\lambda) &= \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \int_1^{+\infty} du (-1) \int_{\mathcal{C}} \frac{ds}{2i \sin(\pi s)} \frac{\lambda^s}{\Gamma(1+s)} u^s \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} \\ &= - \frac{\Gamma(\gamma + \beta)}{\Gamma(\gamma)\Gamma(\beta)} \int_{\mathcal{C}} \frac{ds}{2i \sin(\pi s)} \frac{\lambda^s}{\Gamma(1+s)} \int_1^{+\infty} u^s \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} \\ &= - \int_{\mathcal{C}} \frac{ds}{2i \sin(\pi s)} \frac{\lambda^s}{\Gamma(1+s)} \frac{\Gamma(\gamma + \beta)\Gamma(\gamma - s)}{\Gamma(\gamma + \beta - s)\Gamma(\gamma)} \end{aligned} \quad (\text{D4})$$

where here the contour of integration \mathcal{C} is a vertical line $\mathcal{C} = -a + i\mathbb{R}$ with $0 < a < 1$. In this way, one can invert the different integrals and the results only contains complex moments $\overline{u^s}$ in a region where they are defined. The relation between $g(\lambda)$ and $g_{mom}(\lambda)$ now appears clearly by comparing (D3) and (D4): $g(\lambda)$ can formally be obtained by rewriting the sum appearing in $g_{mom}(\lambda)$ as a Mellin-Barnes transform. Note that when closing the contour of integration \mathcal{C} on the $\text{Re}(s) > 0$ half-plane in D4, one obtains two types of poles. A first series coming from the sine function that reproduces the series that defines $g_{mom}(\lambda)$, as well as a second series of terms of the form $\lambda^{\gamma+n}$ with $n \in \mathbb{N}$ coming from the poles of $\Gamma(\gamma - s)$: $g(\lambda)$ is not an analytic function of λ . Rewriting the sum appearing in

$g_{mom}(\lambda)$ as a Mellin-Barnes integral thus allows us in some way to retrieve the missing, non-analytic terms that are present in the Laplace transform. In the main text we use the same prescription to go from a Fredholm determinant formula for $g_{t,x}^{mom}(u)$ to a formula for $g_{t,x}(u)$ by rewriting the sum over m appearing in the expression of the kernel (68) as a Mellin-Barnes type integral in (69). Notice that in (68), the sum over m runs from 1 to ∞ , and the associated integral written in (69) is thus chosen as a line that passes through the right of 0 (a trivial modification of the case studied here), and to the left of γ to avoid crossing a pole.

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